Generalization of Single-Valued Mapping with Contraction Concept in Complex-Valued Metric Space

Nisha Sharma
Department of Mathematics
Manav Rachna International University Faridabad

Mamta Rani
Department of Mathematics
Pt. JLN Govt. College Faridabad

Abstract

Owning the concept of complex-valued metric spaces, which was introduced by Azam et al. [4], who introduced the new concept and established a common fixed point result in the context of complex-valued metric spaces. In this paper, we generalized the concept of chatterjea [14] contraction mapping for single-valued mapping on the complex-valued metric spaces.
Keywords: Complex-valued metric space, single-valued mappings, Contraction mappings, Common fixed point, lower bound and greater lower bound (g.l.b)
MSC: 46S40, 47H10, 54H25

I. INTRODUCTION

It is well known fact that the mathematical results regarding fixed points of contraction-type mappings are very useful for determining the existence and uniqueness of solutions to various mathematical models. The theory of fixed points has been developed, regarding the results to finding the fixed points self and nonself over the last 51 years.

Many authors have proved fixed point results in the different kind of generalization in complex-valued metric spaces. Nadler [9] and Markin [8] was initiated the study of fixed points for multi-valued contraction mappings. Azam et al. [4] introduced the concept of complex-valued metric space and obtained sufficient conditions for the existence of common fixed points. Very recently, Ahmad et al. [2] obtained some new fixed point results for multi-valued mappings in the setting of complex-valued metric spaces. Some fixed point results by generalizing the contractive conditions in the context of complex-valued metric spaces was established by Sitthikul and Saejung [13] and Klin-eam and Suanoom [7]. The results presented in this paper substantially extend the results given by chatterjea et.al [14] for the multi-valued mappings.

II. PRELIMINARIES

Let C be the set of complex numbers and $z_1, z_2 \in C$. Define a partial order $\precsim$ C on as follows:
$z_1 \precsim z_2$ if and only if $Re(z_1) \le Re(z_2)$, $Im(z_1) \le Im(z_2)$.
It follows that $z_1 \precsim z_2$
if one of the following conditions is satisfied:
1) $Re(z_1) = Re(z_2)$, $Im(z_1) < Im(z_2)$,
2) $Re(z_1) < Re(z_2)$, $Im(z_1) = Im(z_2)$,
3) $Re(z_1) < Re(z_2)$, $Im(z_1) < Im(z_2)$,
4) $Re(z_1) = Re(z_2)$, $Im(z_1) = Im(z_2)$,

In particular, we will write $z_1 \precnsim z_2$ if $z_1 \neq z_2$ and one of (i),(ii) and (iii) is satisfied and we will write $z_1 \prec z_2$ if only (iii) is satisfied. Note that

$$0 \precsim z_1 \precnsim z_2 \Rightarrow |z_1| < |z_2|,$$
$$z_1 \precsim z_2 \ , \ z_2 \prec z_3 \Rightarrow z_1 \prec z_3 .$$

A. *Definition 1.3*

Let X is a nonempty set. Suppose that the mapping $d : X \times X \to C$ satisfies:
1) $0 \precsim d(x,y)$ for all $x,y \in X$ and $d(x,y) = 0$ if and only if $x = y$;
2) $d(x,y) = d(y,x)$ for all $x,y \in X$
3) $d(x,y) \precsim d(y,x) + d(z,y)$ for all $x,y,z \in X$.

Then d is called a complex-valued metric on X, and (X,d) is called a complex-valued metric space.
A point $x \in X$ is called an interior point of a set $A \subseteq X$ whenever there exists $0 \prec r \in C$ such that
$B(x,r) = \{y \in X : d(x,y) \prec r\} \subseteq A$.

A point $x \in X$ is called a limit point of A whenever, for every $0 \prec r \in C$
$B(x,r) \cap (A\setminus\{x\}) \neq \phi$.
A is called open whenever each element of *A* is an interior point of A. Moreover, a subset $B \subseteq X$ is called closed whenever each limit point of B belongs to B.

Let $\{x_n\}$ be a sequence in X and $x \in X$. If for every $c \in C$ with $0 \prec c$ there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n,x) \prec c$, then $\{x_n\}$ is said to be convergent, $\{x_n\}$ converges to *x* and *x* is the limit point of $\{x_n\}$. We denote this by $\lim_{n\to\infty} x_n = x$, or $x_n \to x$ as $n\to\infty$.

If for every $c \in C$ with $0 \prec c$ there is $n_0 \in N$ such that for all $n > n_0$, $d(x_n,x_{n+m}) \prec c$, where $m \in N$, then $\{x_n\}$ is called a Cauchy sequence in (X,d). If every Cauchy sequence is convergent in (X,d), then (X,d) is called a complete complex-valued metric space. We require the following lemmas.

1) *Lemma 1.4*

[4] Let (X,d) be a complex-valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converges to x if and only if $|d(x_n,x)| \to 0$ as $n\to\infty$.

2) *Lemma 1.5*

[4] Let *(X,d)* be a complex-valued metric space and let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $|d(x_n,x_{n+m})| \to 0$ as $n\to\infty$, where $m \in N$.

B. *Definition 1.6 [4]*

Let (X,d) be a complex-valued metric space.
We denote the family of nonempty, closed and bounded subsets of a complex valued metric space by CB(X).
From now on, we denote $s(z_1) = \{z_2 \in C: z_1 \preceq z_2\}$ for $z_1 \in C$, and
$s(a,B) = \bigcup_{b\in B} s(d(a,b)) = \bigcup_{b\in B} \{z \in C: d(a,b) \preceq z\}$ for $a \in X$ and $B \in CB(X)$.
For $A,B \in CB(X)$, we denote

$$s(A,B) = \left(\bigcap_{a\in A} s(a,B)\right) \cap \left(\bigcap_{b\in B} s(b,A)\right).$$

C. *Definition 1.7*

[2] Let (X,d) be a complex-valued metric space. Let $T: X\to CB(X)$ be a multi-valued map. For $x \in X$ and $A \in CB(X)$, define
$W_x(A) = \{d(x,a) : a \in A\}$.
Thus, for $x,y \in X$,
$W_x(T_y) = \{d(x,u) : u \in Ty\}$.

D. *Definition 1.8*

[2] Let (X,d) be a complex-valued metric space. A subset A of X is called bounded from below if there exists some $z \in X$ such that $z \preceq a$ for all $a \in A$.

E. *Definition 1.9*

[2] Let (X,d) be a complex - valued metric space. A multi - valued mapping $F: X \to 2^C$ is called bounded from below if for each $x \in X$ there exists $z_x \in C$ such that $z_x \preceq u$ for all $u \in Fx$.

F. *Definition 1.10*

[2] Let (X,d) be a complex-valued metric space. The multi - valued mapping $T:X\to CB(X)$ is said to have the lower bound property (l.b property) on (X,d) if for any $x\in X$, the multi-valued mapping $F_x : X\to 2^C$ defined by

$$F_x(y) = W_x(Ty)$$

is bounded from below. That is, for $x,y \in X$, there exists an element $l_x(Ty) \in C$ such that

$$l_x(Ty) \preceq u$$

for all $u \in W_x(Ty)$, where $l_x(Ty)$ is called a lower bound of T associated with (x,y).

G. *Definition 1.11*

[2] Let (X,d) be a complex-valued metric space. The multi-valued mapping $T: X\to CB(X)$ is said to have the greatest lower bound property (g.l.b property) on (X,d) if a greatest lower bound of $W_x(Ty)$ exists in C for all $x,y \in X$. We denote d(x,Ty)by the g.l.b of $W_x(Ty)$. That is,
$d(x,Ty) = \inf\{d(x,u) : u \in Ty\}$.

III. MAIN RESULT

Theorem 1.12. Let (X, d) be a complete complex-valued metric space and let T: X → CB (X) be single-valued mapping with g.l.b property such that

$$(1.1)\quad \alpha\left\{\frac{d(x,Tx)d(x,Ty)+d(y,Ty)d(y,Tx)}{d(x,Ty)+d(y,Tx)}\right\}+\beta\left\{\frac{d(x,Tx)d(y,Ty)+d(x,Tx)d(y,Tx)}{d(y,Ty)}\right\}$$

$$+\gamma\left\{\frac{d(x,Tx)d(y,Ty)+d^2(Tx,y)}{d(Tx,Ty)+d(y,Tx)}\right\}+\delta\left\{\frac{d(x,Tx)d(x,Ty)+d(y,Tx)}{d(x,Ty)+d(x,Tx)}\right\}$$

$$+\lambda\left\{\frac{d(x,y)d(Tx,Ty)+d(y,Tx)}{d(Tx,Ty)+d(y,Tx)}\right\}\in s(Tx,Ty)$$

For all $x, y \in X$ and $0 \le \alpha+\beta+\gamma+\delta+\lambda < 1$. Then T has a common fixed point.

Proof : Let $x_0 \in X$ and $x_1 \in Tx_0$ from (1.1), we get

$$\alpha\left\{\frac{d(x_0,Tx_0)d(x_0,Tx_1)+d(x_1,Tx_1)d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}+\beta\left\{\frac{d(x_0,Tx_0)d(x_1,Tx_1)+d(x_0,Tx_0)d(x_1,Tx_0)}{d(x_1,Tx_1)}\right\}$$

$$+\gamma\left\{\frac{d(x_0,Tx_0)d(x_1,Tx_1)+d^2(Tx_0,x_1)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}+\delta\left\{\frac{d(x_0,Tx_0)d(x_0,Tx_1)+d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}$$

$$+\lambda\left\{\frac{d(x_0,x_1)d(Tx_0,Tx_1)+d(x_1,Tx_0)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}\in s(Tx_0,Tx_1)\in\bigcap_{x\in Sx_0} s(x,Tx_1)$$

That is

$$\alpha\left\{\frac{d(x_0,Tx_0)d(x_0,Tx_1)+d(x_1,Tx_1)d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}+\beta\left\{\frac{d(x_0,Tx_0)d(x_1,Tx_1)+d(x_0,Tx_0)d(x_1,Tx_0)}{d(x_1,Tx_1)}\right\}$$

$$+\gamma\left\{\frac{d(x_0,Tx_0)d(x_1,Tx_1)+d^2(Tx_0,x_1)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}+\delta\left\{\frac{d(x_0,Tx_0)d(x_0,Tx_1)+d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}$$

$$+\lambda\left\{\frac{d(x_0,x_1)d(Tx_0,Tx_1)+d(x_1,Tx_0)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}\in s(x,Tx_1)\ \text{For all}\ x\in Sx_0$$

Since $x_1 \in Tx_0$, so we have

$$\alpha\left\{\frac{d(x_0,Tx_0)\,d(x_0,Tx_1)+d(x_1,Tx_1)\,d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}+\beta\left\{\frac{d(x_0,Tx_0)\,d(x_1,Tx_1)+d(x_0,Tx_0)\,d(x_1,Tx_0)}{d(x_1,Tx_1)}\right\}$$

$$+\gamma\left\{\frac{d(x_0,Tx_0)\,d(x_1,Tx_1)+d^2(Tx_0,x_1)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}+\delta\left\{\frac{d(x_0,Tx_0)\,d(x_0,Tx_1)+d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}$$

$$+\lambda\left\{\frac{d(x_0,x_1)\,d(Tx_0,Tx_1)+d(x_1,Tx_0)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}\in s(x_1,Tx_1),$$

$$=\bigcup_{x\in Tx_1} s(d(x_1,x))$$

So there exists some $x_2 \in Tx_1$ such that

$$\alpha\left\{\frac{d(x_0,Tx_0)\,d(x_0,Tx_1)+d(x_1,Tx_1)\,d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}+\beta\left\{\frac{d(x_0,Tx_0)\,d(x_1,Tx_1)+d(x_0,Tx_0)\,d(x_1,Tx_0)}{d(x_1,Tx_1)}\right\}$$

$$+\gamma\left\{\frac{d(x_0,Tx_0)\,d(x_1,Tx_1)+d^2(Tx_0,x_1)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}+\delta\left\{\frac{d(x_0,Tx_0)\,d(x_0,Tx_1)+d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}$$

$$+\lambda\left\{\frac{d(x_0,x_1)\,d(Tx_0,Tx_1)+d(x_1,Tx_0)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}\in s(d(x_1,x_2))$$

That is,

$$d(x_1,x_2)\preceq\alpha\left\{\frac{d(x_0,Tx_0)\,d(x_0,Tx_1)+d(x_1,Tx_1)\,d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}+\beta\left\{\frac{d(x_0,Tx_0)\,d(x_1,Tx_1)+d(x_0,Tx_0)\,d(x_1,Tx_0)}{d(x_1,Tx_1)}\right\}$$

$$+\gamma\left\{\frac{d(x_0,Tx_0)\,d(x_1,Tx_1)+d^2(Tx_0,x_1)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}+\delta\left\{\frac{d(x_0,Tx_0)\,d(x_0,Tx_1)+d(x_1,Tx_0)}{d(x_0,Tx_1)+d(x_1,Tx_0)}\right\}$$

$$+\lambda\left\{\frac{d(x_0,x_1)\,d(Tx_0,Tx_1)+d(x_1,Tx_0)}{d(Tx_0,Tx_1)+d(x_1,Tx_0)}\right\}$$

By using the greatest lower property (g.l.b. property) of T , we get.

$$d(x_1,x_2) \preceq \alpha\left\{\frac{d(x_0,x_1)d(x_0,x_2)+d(x_1,x_2)d(x_1,x_1)}{d(x_0,x_2)+d(x_1,x_1)}\right\}+\beta\left\{\frac{d(x_0,x_1)d(x_1,x_2)+d(x_0,x_1)d(x_1,x_1)}{d(x_1,x_2)}\right\}$$

$$+\gamma\left\{\frac{d(x_0,x_1)d(x_1,x_2)+d^2(x_1,x_1)}{d(x_1,x_2)+d(x_1,x_1)}\right\}+\delta\left\{\frac{d(x_0,x_1)d(x_0,x_2)+d(x_1,x_1)}{d(x_0,x_2)+d(x_1,x_1)}\right\}$$

$$+\lambda\left\{\frac{d(x_0,x_1)d(x_1,x_2)+d(x_1,x_1)}{d(x_1,x_2)+d(x_1,x_1)}\right\}$$

Which implies that

$$d(x_1,x_2) \preceq \alpha d(x_0,x_1)+\beta d(x_0,x_1)+\gamma d(x_0,x_1)+\delta d(x_0,x_1)+\lambda d(x_0,x_1)$$

$$d(x_1,x_2) \preceq (\alpha+\beta+\gamma+\delta+\lambda)d(x_0,x_1)$$

Similarly

$$d(x_2,x_3) \preceq (\alpha+\beta+\gamma+\delta+\lambda)^2 d(x_0,x_1)$$

$$d(x_3,x_4) \preceq (\alpha+\beta+\gamma+\delta+\lambda)^3 d(x_0,x_1)$$

| |

| |

Inductively, we can construct a sequence $\{x_n\}$ in X such that for n = 1,2 ………..,

$|d(x_n,x_{n+1})| \le (\alpha+\beta+\gamma+\delta+\lambda)^n |d(x_0,x_1)|$ with $(\alpha+\beta+\gamma+\delta+\lambda)<1, x_{2n+1} \in Tx_{2n}$ and $x_{2n+2} \in Tx_{2n+1}$.

Now for $m > n$, we get

$$|d(x_n,x_m)| \le |d(x_n,x_{n+1})|+|d(x_{n+1},x_{n+2})|+-----+|d(x_{m-1},x_m)| \le \left[a^n+a^{n+1}+...+a^{m-1}\right]|d(x_0,x_1)|$$

Where $a=\alpha+\beta+\gamma+\delta+\lambda$

$$|d(x_n,x_m)| \le \left[\frac{a^n}{1-a}\right]|d(x_0,x_1)|, \quad since\ 0 \le a < 1.$$

And so $|d(x_n,x_m)| \to 0$ as m, n $\to \infty$.

This implies that $\{x_n\}$ is a Cauchy Sequence in X. Since X is complete, so there exists $v \in X$ such that $x_n \to v$ as $n \to \infty$.

We now show that $v \in Tv$.

From (1.1),we have

$$\alpha\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tv)d(v,Tx_{2n})}{d(x_{2n},Tv)+d(v,Tx_{2n})}\right\}$$

$$+\beta\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d(x_{2n},Tx_{2n})d(v,Tx_{2n})}{d(v,Tv)}\right\}$$

$$+\gamma\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d^{2}(Tx_{2n},v)}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\}+\delta\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tx_{2n})}{d(x_{2n},Tx_{2n})+d(v,Tx_{2n})}\right\}$$

$$+\lambda\left\{\frac{d(x_{2n},v)d(Tx_{2n},Tv)+d(v,Tx_{2n})}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\}\in s(Tx_{2n},Tv)$$

$$\in\bigcap_{x\in Sx_{2n}} s(x,Tv)$$

and so

$$\alpha\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tv)d(v,Tx_{2n})}{d(x_{2n},Tv)+d(v,Tx_{2n})}\right\}$$

$$+\beta\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d(x_{2n},Tx_{2n})d(v,Tx_{2n})}{d(v,Tv)}\right\}$$

$$+\gamma\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d^{2}(Tx_{2n},v)}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\}+\delta\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tx_{2n})}{d(x_{2n},Tx_{2n})+d(v,Tx_{2n})}\right\}$$

$$+\lambda\left\{\frac{d(x_{2n},v)d(Tx_{2n},Tv)+d(v,Tx_{2n})}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\}\in s(x,Tv)\ \text{ for all } x\in Tx_{2n}$$

Since $x_{2n+1}\in Tx_{2n}$, so we have

$$\alpha\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tv)d(v,Tx_{2n})}{d(x_{2n},Tv)+d(v,Tx_{2n})}\right\}$$

$$+\beta\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d(x_{2n},Tx_{2n})d(v,Tx_{2n})}{d(v,Tv)}\right\}$$

$$+\gamma\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d^{2}(Tx_{2n},v)}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\}+\delta\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tx_{2n})}{d(x_{2n},Tx_{2n})+d(v,Tx_{2n})}\right\}$$

$$+\lambda\left\{\frac{d(x_{2n},v)d(Tx_{2n},Tv)+d(v,Tx_{2n})}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\}\in s(x_{2n+1},Tv)$$

$$\in s\left(x_{2n+1}, Tv\right) = \bigcup_{u' \in Tv} s\left(d\left(x_{2n+1}, u'\right)\right)$$

so there exists some $v_n \in Tv$ such that

$$\alpha\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tv)d(v,Tx_{2n})}{d(x_{2n},Tv)+d(v,Tx_{2n})}\right\} +\beta\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d(x_{2n},Tx_{2n})d(v,Tx_{2n})}{d(v,Tv)}\right\}$$

$$+\gamma\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d^2(Tx_{2n},v)}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\} +\delta\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tx_{2n})}{d(x_{2n},Tx_{2n})+d(v,Tx_{2n})}\right\}$$

$$+\lambda\left\{\frac{d(x_{2n},v)d(Tx_{2n},Tv)+d(v,Tx_{2n})}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\} \in s\left(d\left(x_{2n+1}, v_n\right)\right),$$

That is,

$$d\left(x_{2n+1}, v_n\right) \preceq \alpha\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tv)d(v,Tx_{2n})}{d(x_{2n},Tv)+d(v,Tx_{2n})}\right\}$$

$$+\beta\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d(x_{2n},Tx_{2n})d(v,Tx_{2n})}{d(v,Tv)}\right\}$$

$$+\gamma\left\{\frac{d(x_{2n},Tx_{2n})d(v,Tv)+d^2(Tx_{2n},v)}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\} +\delta\left\{\frac{d(x_{2n},Tx_{2n})d(x_{2n},Tv)+d(v,Tx_{2n})}{d(x_{2n},Tx_{2n})+d(v,Tx_{2n})}\right\}$$

$$+\lambda\left\{\frac{d(x_{2n},v)d(Tx_{2n},Tv)+d(v,Tx_{2n})}{d(Tx_{2n},Tv)+d(v,Tx_{2n})}\right\}$$

By using the greatest lower property (g.l.b. property) of T, we get.

$$d\left(x_{2n+1}, v_n\right) \preceq \alpha\left\{\frac{d\left(x_{2n}, x_{2n+1}\right)d\left(x_{2n}, v\right)+d\left(v, Tv\right)d\left(v, x_{2n+1}\right)}{d\left(x_{2n}, v_n\right)+d\left(v, x_{2n+1}\right)}\right\}$$

$$+\beta\left\{\frac{d\left(x_{2n}, x_{2n+1}\right)d\left(v, Tv\right)+d\left(x_{2n}, x_{2n+1}\right)d\left(v, x_{2n+1}\right)}{d\left(v, v_n\right)}\right\}$$

$$+\gamma\left\{\frac{d\left(x_{2n}, x_{2n+1}\right)d\left(v, v_n\right)+d^2\left(x_{2n+1}, v\right)}{d\left(x_{2n+1}, v_n\right)+d\left(v, x_{2n+1}\right)}\right\}$$

$$+\delta\left\{\frac{d\left(x_{2n}, x_{2n+1}\right)d\left(x_{2n}, v_n\right)+d\left(v, x_{2n+1}\right)}{d\left(x_{2n}, x_{2n+1}\right)+d\left(v, x_{2n+1}\right)}\right\} +\lambda\left\{\frac{d\left(x_{2n}, v\right)d\left(x_{2n+1}, v_n\right)+d\left(v, x_{2n+1}\right)}{d\left(x_{2n+1}, v_n\right)+d\left(v, x_{2n+1}\right)}\right\}$$

By using again the triangular inequality, we get

$d\left(v, v_n\right) \preceq d\left(v, x_{2n+1}\right) + d\left(x_{2n+1}, v_n\right)$.

Then we have

$$d(v,v_n) \preceq d(v,x_{2n+1}) + \alpha\left\{\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},v)+d(v,Tv)\,d(v,x_{2n+1})}{d(x_{2n},v_n)+d(v,x_{2n+1})}\right\}$$

$$+\beta\left\{\frac{d(x_{2n},x_{2n+1})\,d(v,Tv)+d(x_{2n},x_{2n+1})\,d(v,x_{2n+1})}{d(v,v_n)}\right\}$$

$$+\gamma\left\{\frac{d(x_{2n},x_{2n+1})\,d(v,v_n)+d^2(x_{2n+1},v)}{d(x_{2n+1},v_n)+d(v,x_{2n+1})}\right\}$$

$$+\delta\left\{\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},v_n)+d(v,x_{2n+1})}{d(x_{2n},x_{2n+1})+d(v,x_{2n+1})}\right\}+\lambda\left\{\frac{d(x_{2n},v)\,d(x_{2n+1},v_n)+d(v,x_{2n+1})}{d(x_{2n+1},v_n)+d(v,x_{2n+1})}\right\}$$

and we obtain

$$|d(v,v_n)| \le |d(v,x_{2n+1})| + \alpha\left|\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},v)+d(v,Tv)\,d(v,x_{2n+1})}{d(x_{2n},v_n)+d(v,x_{2n+1})}\right|$$

$$+\beta\left|\frac{d(x_{2n},x_{2n+1})\,d(v,Tv)+d(x_{2n},x_{2n+1})\,d(v,x_{2n+1})}{d(v,v_n)}\right|$$

$$+\gamma\left|\frac{d(x_{2n},x_{2n+1})\,d(v,v_n)+d^2(x_{2n+1},v)}{d(x_{2n+1},v_n)+d(v,x_{2n+1})}\right|$$

$$+\delta\left|\frac{d(x_{2n},x_{2n+1})\,d(x_{2n},v_n)+d(v,x_{2n+1})}{d(x_{2n},x_{2n+1})+d(v,x_{2n+1})}\right| +\lambda\left|\frac{d(x_{2n},v)\,d(x_{2n+1},v_n)+d(v,x_{2n+1})}{d(x_{2n+1},v_n)+d(v,x_{2n+1})}\right|$$

By letting $n \to \infty$ in the above inequality , we get $|d(v,v_n)| \to 0$ as $n \to \infty$. By Lemma (1.4), we have $v_n \to v$ as $n \to \infty$. Since Tv is closed, so $v \in Tv$. Thus T has a common fixed point.

References

[1] Abbas, M, Fisher, B, Nazir, T: Well-posedness and periodic point property of mappings satisfying a rational inequality in an ordered complex -valued metric spaces, Number,Funct Anal Optim 32,243-253(2011).
[2] Ahmad, J, klin-eam , C, Azam, A Commom fixed points for multi-valued mappings in complex valued metric spaces with applications Abstr,Appl,Anal, 2013,Article ID 854965(2013).
[3] Akbar Azam, Jamshaid Ahmad and Poom Kumam, Common fixed point theorems for multi-valued mappings in complex-valued metric spaces Journal of Inequalities 10.1186/1029- 242X-2013-578, Springer. 2013.
[4] Azam, A, Fisher, B, Khan M; Corrigendum: 'Common fixed point theorems in complex valued metric spaces';Numer Funct Anal Optim 33(5), 590-600(2012).
[5] Kannan, R: Some results on fixed points,Bull,Calcutta Math,Soc ,60,71-76(1968).
[6] Khan MS: A fixed point theorem for metric spaces. Rend. Ist. Mat. Univ. Trieste 1976, 8:69–72.
[7] Klin-eam C, Suanoom C: Some common fixed point theorems for generalized-contractive - type mappings on complex valued metric spaces. Abstr. Appl. Anal. 2013. 2013: Article ID 604215 10.1155/2013/604215.
[8] Markin, JT: Continuous dependence of fixed point sets Proc Am Math, Soc 38,545-547 (1973).
[9] Nadler, SB Jr.: Multi-valued contraction mappings,Pac. J Math 30,475-478 (1969).

[10] Rouzkard F, Imdad M: Some common fixed point theorems on complex valued metric spaces. Comput. Math. Appl. 2012. 10.1016/j.camwa.2012.02.063.
[11] Sintunavarat W, Kumam P: Generalized common fixed point theorems in complex valued metric spaces and applications. J. Inequal. Appl.2012. 2012: Article ID 84.
[12] Sintunavarat W, Cho YJ, Kumam P: Urysohn integral equations approach by common fixed points in complex valued metric spaces. Adv. Differ. Equ. 2013. 2013: Article ID 49.
[13] Sitthikul K, Saejung S: Some fixed point theorems in complex valued metric spaces. Fixed Point Theory Appl. 2012. 2012: Article ID 189.
[14] Suzuki-Type Generalization of Chatterjea Contraction Mappings on Complete Partial Metric space Hindawi Publishing Corporation, journal of operators, vol. 2013, Article ID 923843, 10.1155/2013/928843.