An Approximate Solution of the Two Dimensional Unsteady Flow due to Normally Expanding or Contracting Parallel Plates

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Abstract

The flow of a viscous incompressible fluid between two parallel plates due to the normal motion of the plates for the two-dimensional flow is investigated. The governing nonlinear equations and their associated boundary conditions are transformed into linear differential equation using quasilinearization technique. The solution of the problem is obtained by Quintic spline collocation method. Graphical results are presented to investigate the influence of the squeeze number on the velocity, skin friction, and pressure gradient. The validity of our solution is verified by analytic results obtained by homotopy perturbation method (HPM).

Keywords: Fourth Order Ordinary Differential Equation, Quasilinearization, Quintic Spline Collocation, Upper Triangular Matrix, Squeeze Number

I. INTRODUCTION

Fluid flow between two parallel surfaces has received much attention because of its importance in many fields of science and engineering. Squeezing flow between parallel walls occurs lots of application in industrial application. Moving pistons in engines, hydraulic brakes, chocolate filler and many other devices are based on the principle of flow between contracting domains. Stefan [1] published a classical paper on squeezing flow by using lubrication approximation. In 1886, Reynolds [2] obtained a solution for elliptic plates, and Archibald [3] studies this problem for rectangular plates. The numerical study of the asymmetric flow has been carried out by Lai et al. [4-5]. Numerical solution of a system of fourth order by Al- Said, E.A. and Noor, M.A.[6] and quintic spline solution by Aziz, T. and Khan, A [7].Saeed et al[8] examined the problem of unsteady flows due to normally expanding and contracting parallel plates analytically by using HPM. Solution of higher order differential equations using numerical method in [9-16].

The governing equations here are highly nonlinear differential equations, which are solved by using the Quintic spline collocation method. In this way, the paper has been organized as follows. In section 2, the problem statement and mathematical formulation are presented. In section 3, we discuss the Quintic spline collocation method. Approximate solution for the governing equations are obtained using quintic spline method in section-4, Section-5 contains the results and discussion. The conclusions are summarized in section 6.

II. FORMULATION OF THE PROBLEM

Let the position of the two plates be at $z = \pm l(1-\alpha t)^{1/2}$, where l is the position at time t = 0 as shown in below figure.

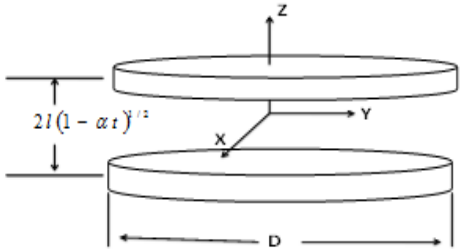

Fig. 1: Schematic diagram of the problem.

We assume that length l in two-dimensional case or the diameter D (in the axisymmetric case) is much larger than the gap width $2z$ at any time such that the end effect can be neglected. When α is positive, the two plates are squeezed until they touch

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at $t = 1/\alpha$. When α is negative, the two plates are separated. Let u, v and w be the velocity component in the x, y and z directions. For two-dimensional flow, Wang introduced the following transform [8].

$$u = \frac{\alpha x}{[2(1-\alpha t)]} f'(n),$$
$$w = \frac{-\alpha l}{\left[2(1-\alpha t)^{1/2}\right]} f(n), \qquad (1)$$

Where

$$n = \frac{z}{\left[l(1-\alpha t)^{1/2}\right]}. \qquad (2)$$

Substitute (1) into unsteady two-dimensional Navier-Stokes equations yields non-liner ordinary differential equation in form:

$$f'''' + S\left\{-nf''' - 3f'' - f'f'' + ff'''\right\} = 0. \qquad (3)$$

Where $S = al^2/2v$ (squeeze number) is the nondimensional parameter. The flow is characterized by this parameter. The boundary conditions are such that on the plates the lateral velocities are zero and the normal velocity is equal to the velocity of the plates, that is,

$$f(0) = 0, \qquad f''(0) = 0, \qquad f(1) = 1, \qquad f'(1) = 0, \qquad (4)$$

Similarly Wang's Transform [36] for axisymmetric flow are,

$$u = \frac{\alpha x}{[4(1-\alpha t)]} f'(n), \qquad (5)$$
$$v = \frac{\alpha y}{[4(1-\alpha t)]} f'(n),$$
$$w = \frac{-\alpha l}{\left[2(1-\alpha t)^{1/2}\right]} f(n),$$

Using transform (5), unsteady axisymmetric Navier- Stokes equation reduce to

$$f'''' + S\left\{-nf''' - 3f'' - ff'''\right\} = 0, \qquad (6)$$

Subject to boundary condition (4) ,Consequently we solve nonlinear ordinary differential equation

$$f'''' + S\left\{-nf''' - 3f'' - \beta f'f'' + ff'''\right\} = 0 \quad , \qquad (7)$$

Where

$$\beta = \begin{cases} 0, & Axisymmtric, \\ 1, & Two-\dim ensional, \end{cases} \qquad (8)$$

And subject to boundary conditions (4).

III. QUINTIC SPLINE COLLOCATION METHOD

Consider equally spaced knots of partition π: $a = x_0 < x_1 < x_2 < ..... < x_n = b$ on $[a,b]$. The Quintic Spline[15] is

$$s(x) = a_0 + b_0(x-x_0) + \frac{1}{2}c_0(x-x_0)^2 + \frac{1}{6}d_0(x-x_0)^3 + \frac{1}{24}e_0(x-x_0)^4 + \frac{1}{120}\sum_{k=0}^{n-1} F_k(x-x_k)_+^5 \qquad (9)$$

Where the powers function $(x-x_k)_+$ is defined as

$$(x-x_k)_+ = \begin{cases} x-x_k, & x > x_k \\ 0, & x \le x_k \end{cases} \qquad (10)$$

Consider a fourth -order linear boundary value problem of the form

$$y^{iv} + p(x)y'''(x) + q(x)y''(x) + r(x)y'(x) + t(x)y(x) = m(x), a \le x \le b, \qquad (11)$$

Subject to boundary conditions

$$\alpha_0 y_0 + \beta_0 y_n' + \gamma_0 y_n'' + \delta_0 y_n''' = \eta_0,$$
$$\alpha_1 y_0' + \beta_1 y_n + \gamma_1 y_n'' + \delta_1 y_n''' = \eta_1,$$
$$\alpha_2 y_0'' + \beta_2 y_n + \gamma_2 y_n'' + \delta_2 y_n''' = \eta_2,$$
$$\alpha_3 y_0''' + \beta_3 y_n + \gamma_3 y_n'' + \delta_3 y_n''' = \eta_3, \qquad (12)$$

Where $y(x)$, $p(x)$, $q(x)$, $r(x)$, $t(x)$ and $m(x)$ are continuous function defined in the interval $x \in [a,b]$; $\eta_0, \eta_1, \eta_2, \eta_3$ are finite real constants.

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Let (9) be an approximation solution of (11), where $a_0, b_0, c_0, d_0, e_0, F_0, F_1, ..., F_{n-1}$ are real coefficient to be determined. Let $x_0, x_1, ..., x_n$ be $n+1$ grid points in the interval $[a, b]$, so that

$$x_i = a + ih,\ i = 0,1,2,....., n;\quad x_0 = a,\quad x_n = b,\qquad h = \frac{b-a}{n}. \tag{13}$$

It is required that approximate solution (9) satisfies the differential equation at the point $x = x_i$.

Putting (9) with its successive derivatives in (11), we obtain the collocation equations as follows:

$$\begin{aligned}
&\sum_{k=0}^{n-1} F_k \{(x_i - x_k)_+ + \frac{1}{2} p(x_i)(x_i - x_k)_+^2 + \frac{1}{6} q(x_i)(x_i - x_k)_+^3 + \frac{1}{24} r(x_i)(x_i - x_k)_+^4 + \frac{1}{120} t(x_i)(x_i - x_k)_+^5 \} \\
&+ e_0 \{1 + p(x_i)(x_i - x_0) + \frac{1}{2} q(x_i)(x_i - x_0)^2 + \frac{1}{6} r(x_i)(x_i - x_0)^3 + \frac{1}{24} t(x_i)(x_i - x_0)^4 \} \\
&+ d_0 \{ p(x_i) + q(x_i)(x_i - x_0) + \frac{1}{2} r(x_i)(x_i - x_0)^2 + \frac{1}{6} t(x_i)(x_i - x_0)3 \} \\
&+ c_0 \left\{ q(x_i) + r(x_i)(x_i - x_0) + \frac{1}{2} t(x_i)(x_i - x_0)^2 \} \right\} \\
&+ b_0 \{ r(x_i) + t(x_i)(x_i - x_0) + a_0 (t(x_i)) \} = m(x_i), i = 0,1,2,..... .., n.
\end{aligned} \tag{14}$$

From boundary conditions,

$$\begin{aligned}
&\sum_{k=0}^{n-1} F_k \left(\frac{\delta_0}{2}(b - x_k)_+^2 + \frac{\gamma_0}{6}(b - x_k)_+^3 \right) + e_0 \left(\delta_0 (b-a) + \frac{\gamma_0}{2}(b-a)^2 \right) \\
&+ d_0 \left(\delta_0 + \gamma_0 (b-a) \right) + c_0(\gamma_0) + b_0(\beta_0) + a_0(\alpha_0) = \eta_{0,} \\
&\sum_{k=0}^{n-1} F_k \left(\frac{\delta_1}{2}(b - x_k)_+^2 + \frac{\gamma_1}{6}(b - x_k)_+^3 + \frac{\beta_1}{120}(b - x_k)_+^5 \right) + e_0 \left(\delta_1 (b-a) + \frac{\gamma_1}{2}(b-a)^2 + \frac{\beta_1}{24}(b-a)^4 \right) \\
&+ d_0 \left(\delta_1 + \gamma_1 (b-a) + \frac{\beta_1}{6}(b-a)^3 \right) + c_0(\gamma_1 + \frac{\beta_1}{2}(b-a)^2) + b_0(\beta_1 (b-a) + \alpha_1) + a_0(\beta_1) = \eta_{1,} \\
&\sum_{k=0}^{n-1} F_k \left(\frac{\delta_2}{2}(b - x_k)_+^2 + \frac{\gamma_2}{6}(b - x_k)_+^3 + \frac{\beta_2}{120}(b - x_k)_+^5 \right) + e_0 \left(\delta_2 (b-a) + \frac{\gamma_2}{2}(b-a)^2 + \frac{\beta_2}{24}(b-a)^4 \right) \\
&+ d_0 \left(\delta_2 + \gamma_2 (b-a) + \frac{\beta_2}{6}(b-a)^3 \right) + c_0(\gamma_2 + \frac{\beta_2}{2}(b-a)^2) + b_0(\beta_2 (b-a) + \alpha_2) + a_0(\beta_2) = \eta_{2,} \\
&\sum_{k=0}^{n-1} F_k \left(\frac{\delta_3}{2}(b - x_k)_+^2 + \frac{\gamma_3}{6}(b - x_k)_+^3 + \frac{\beta_3}{120}(b - x_k)_+^5 \right) + e_0 \left(\delta_3 (b-a) + \frac{\gamma_3}{2}(b-a)^2 + \frac{\beta_3}{24}(b-a)^4 \right) \\
&+ d_0 \left(\delta_3 + \gamma_3 (b-a) + \frac{\beta_3}{6}(b-a)^3 \right) + c_0(\gamma_3 + \frac{\beta_3}{2}(b-a)^2) + b_0(\beta_3 (b-a) + \alpha_3) + a_0(\beta_3) = \eta_{3,}
\end{aligned} \tag{15}$$

Using the power function $(x - x_k)_+$ in the above equations, a system of $n+5$ linear equations in $n+5$ unknowns $a_0, b_0, c_0, d_0, e_0, F_0, F_1, ..., F_{n-1}$ is thus obtained. This system can be written in matrix vector form as follows: $AX = B$.

Where $X = [F_{n-1}, F_{n-2,}, ..., F_2, F_1, F_0, e_0, d_0, c_0, b_0, a_0]^T, B = [\eta_3, \eta_2, \eta_1, \eta_0, m(x_n), m(x_{n-1}), ..., m(x_0)]^T$.

The coefficient matrix A is an upper triangular Hessen berg matrix with single lower sub diagonal, principal and upper diagonal having non zero element, because of this nature of a matrix A, the determination of the required quantities becomes simple and consumes less time. The values of these constants ultimately yield the quintic spline s(x) in (9).

In case of nonlinear boundary value problem, the equations can be converted into linear form using Quasilinearization method (Bellman and Kalaba [14]), and hence this method can be used as iterative method. The procedure to obtain a spline approximation of y_i (i=0, 1, 2,…., j, where j denotes the number of iteration) by an iterative method starts with fitting a curve satisfying the end conditions and this curve design as y_i. We obtain the successive iteration y_i 's with the help of the algorithm described as above till desired accuracy.

IV. SOLUTION BY USING COLLOCATION METHOD

In order to solve equation (3) using conditions (4), we use quasilinearization technique to convert into linear form.

$$f_i^{''''} + (-s\eta + sf_i) f_{i+1}^{'''} + (-3s - \beta s f_i) f_{i+1}^{''} + (-\beta s f_i^{''}) f_{i+1}^{'} + s f_i^{'''} f_{i+1} = -\beta s f_i^{''} f_i^{'} + s f_i^{'''} f_i \tag{16}$$

Consider the quintic spline is an approximate solution of equation (16)

$$s(\eta) = a_0 + b_0(\eta - \eta_0) + \frac{1}{2} c_0 (\eta - \eta_0)^2 + \frac{1}{6} d_0 (\eta - \eta_0)^3 + \frac{1}{24} e_0 (\eta - \eta_0)^4 + \frac{1}{120} \sum_{k=0}^{n-1} F_k (\eta - \eta_k)_+^5 \tag{17}$$

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For finding the initial value of fi, ,we assume $f_i(n) = a\eta^3 + b\eta^2 + c\eta + d$ to be the first approximation to start the iterative scheme, which satisfy the given conditions. As discussed in above method, substitute equation (17) with its derivatives in equation (16), we obtain collocation equation as:

$$\begin{aligned}
&\sum_{k=0}^{n-1} F_k[(\eta_i - \eta_k) + \frac{(-s\eta + sf_i)}{2}(\eta_i - \eta_k)^2 + \frac{(-3s - \beta sf_i^{'})}{6}(\eta_i - \eta_k)^3 + \frac{(-\beta sf_i^{''})}{24}(\eta_i - \eta_k)^4 + \frac{(sf_i^{'''})}{120}(\eta_i - \eta_k)^5] \\
&+e_0[1 + (-s\eta + sf_i)(\eta_i - \eta_0) + \frac{(-3s - \beta sf_i^{'})}{2}(\eta_i - \eta_0)^2 + \frac{(-\beta sf_i^{''})}{6}(\eta_i - \eta_0)^3 + \frac{(sf_i^{'''})}{24}(\eta_i - \eta_0)^4] \\
&+d_0[(-s\eta + sf_i) + (-3s - \beta sf_i^{'})(\eta_i - \eta_0) + \frac{(-\beta sf_i^{''})}{2}(\eta_i - \eta_0)^2 + \frac{(sf_i^{'''})}{6}(\eta_i - \eta_0)^3] \\
&+c_0[(-3s - \beta sf_i^{'}) + (-\beta sf_i^{''})(\eta_i - \eta_0) + \frac{(sf_i^{'''})}{6}(\eta_i - \eta_0)^2] \\
&+b_0[(-\beta sf_i^{''}) + (sf_i^{'''})(\eta_i - \eta_0)] \\
&+a_0[(sf_i^{'''})] = -\beta sf_i^{''} f_i^{'} + sf_i^{'''} f_i
\end{aligned} \tag{18}$$

Substitute initial conditions in $s(\eta)$, and we get following equations

$$\begin{aligned}
&a_0 = 0, \\
&c_0 = 0, \\
&a_0 + b_0 + (0.5)c_0 + (0.1666)d_0 + (0.041666)e_0 + (0.008333) f_0 + (0.000273) f_1 + (0.000647) f_2 + (0.00000853) f_3 + (0.00000266) f_4 = 1, \\
&b_0 + c_0 + (0.5)d_0 + (0.16667)e_0 + (0.041666) f_0 + (0.0170666) f_1 + (0.00054) f_2 + (0.0001066) f_3 = 0.
\end{aligned} \tag{19}$$

Solve above equations and substitute constants in (17) and we get the solution for different values of S and $\beta = 1$

A. Graphical Solution of Given Problem

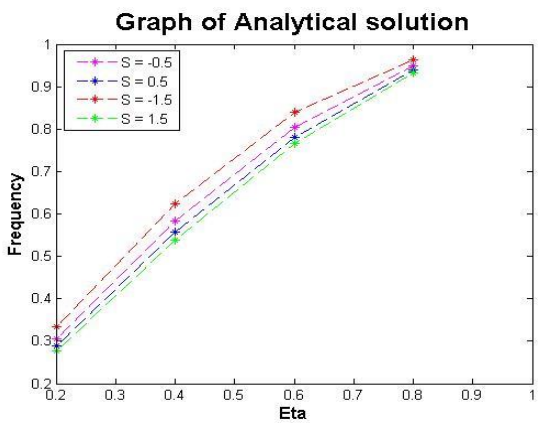

Fig. 2: Graph of Analytical Solution

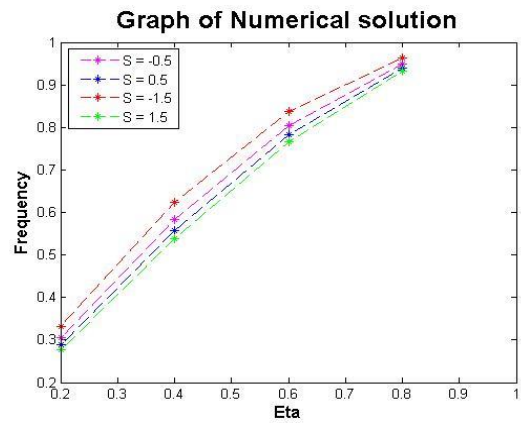

Fig. 3: Graph of Numerical Solution

1) Graph of $f^{'}(\eta)$

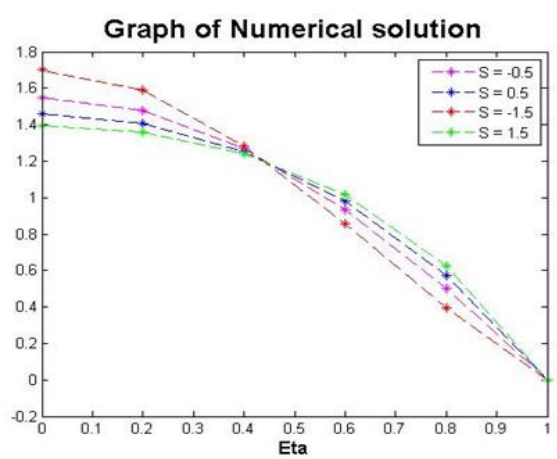

Fig. 4: Graph of $f^{'}(\eta)$

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2) Graph of $f''(1)$

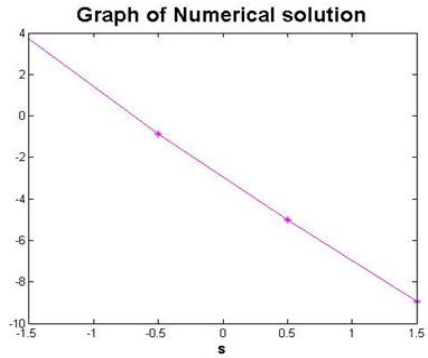

Fig. 5: Graph of $f''(1)$

3) Graph of $f'''(1)$

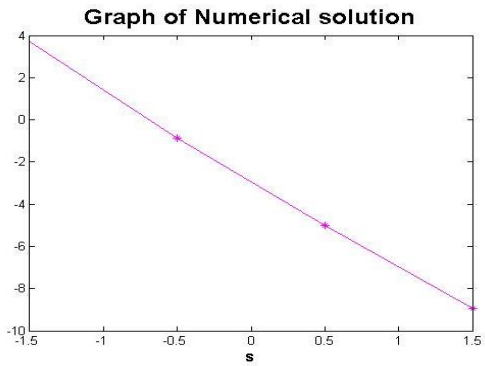

Fig. 6: Graph of $f'''(1)$

V. RESULT AND DISCUSSION

In this section, comparisons of results are made through different squeeze numbers S, All Computations are performed numerically by quintic spline collocation method. The numerical results are agreed with the available analytical solutions. From the figure (ii), the velocity increases due to an increase in S. Figures (ii) & (iii) shows comparison of $f(\eta)$ and $f'(\eta)$ with increase in S. These quantities describe the flow behavior, $f''(1)$ gives skin frication and $f'''(1)$ represents pressure gradient, $f''(1)$ and $f''(1)$ as a function of S are shown in figures (iv) & (v).

VI. CONCLUSION

There are two important goals that we have fulfilled for this study. The first one is to employ successfully the spline collocation method to investigate the behavior of two dimensional squeezing flows between two parallel plates and second is to investigate the influence of the squeeze number on the velocity, skin friction and pressure gradient. Here, the results are compared with HPM. The obtained solutions, in comparison with the numerical solutions, demonstrate remarkable accuracy.

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