Analytical Study on Method of Solving Polynomial Inequalities

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Abstract

A new method to solve the solution set of polynomial inequalities along the real line is conducted. A polynomial function $P_n(x)$ in factored form (of real roots only) with odd multiplicity is emphasized. Let $r_1 < r_2 < \ldots < r_k \in \mathbb{R}$, $m_1 + m_2 + \ldots + m_k = n$ and $n \geq 3$, and let $P_n(x) = (x - r_1)^{m_1}(x - r_2)^{m_2} \ldots (x - r_k)^{m_k}$ if $n$ is even, then the solution set is $\{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{k-1}, r_k) \cup (r_k, \infty) \}$. Thus, when $(x - r_1)^{m_1}(x - r_2)^{m_2} \ldots (x - r_k)^{m_k} \geq 0$, the solution set is $\{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup [r_1, r_2] \cup [r_2, r_3] \cup \ldots \cup [r_{k-1}, r_k] \cup (r_k, \infty) \}$, $i$ is odd number. When $(x - r_1)^{m_1}(x - r_2)^{m_2} \ldots (x - r_k)^{m_k} < 0$, the solution set is $\{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{k-1}, r_k) \cup (r_k, \infty) \}$. Thus, when $(x - r_1)^{m_1}(x - r_2)^{m_2} \ldots (x - r_k)^{m_k} \leq 0$, the solution set is $\{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_1, r_2) \cup \ldots \cup (r_{k-1}, r_k) \cup (r_k, \infty) \}$.

I. INTRODUCTION

The classical problem of solving an $n^{th}$ degree of polynomial equations and inequality has substantially influenced the development of mathematics throughout the centuries and still has several important applications to the theory and practices of present day computing. In particular, the very ideas of abstract thinking and using mathematical notations are the study of problems associated with polynomial equations and inequality. A polynomial, with degree $n$, denoted by $P_n(x)$, is an algebraic expression of the form $a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_1x + a_0$, where $n$ is a non-negative integer and $x$ is a variable and $a_i$'s for $i = 0, 1, 2, \ldots, n$ are the constant coefficients with $a_0 \neq 0$. Polynomial inequality is any inequality that can be put in the form of $P_n(x) > 0$, $P_n(x) < 0$, $P_n(x) \geq 0$ and $P_n(x) \leq 0$ where $P_n(x)$ is a polynomial. A real number $r$ is a solution of the polynomial inequality if, upon the substitution of $r$ for the $x$ in the inequality, the inequality is true. The set of all solutions is called a solution set. When solving a polynomial inequality, just like the polynomial equation, move everything to one side so it is greater than or less than 0. This tells us whether the solutions being looked for are values that give negative or positive results. Next, it will be needed to locate what is known as critical numbers. These are values that create zeros for the polynomial inequality, and when placed in order, divide the number line into “test” sections. This will create test intervals for the inequality which in turn leads to the solution for the polynomial inequality.

II. STATEMENT OF THE PROBLEM

Within systems engineering, behaviour is often reduced to solving for the roots of a polynomial. So, it is essential to design a polynomial to have the correct roots and hence to get the desired behaviour from a system. The concept of polynomial equations and inequalities has many practical applications in engineering, physics and mathematics. Usually it is common to solve polynomial inequalities by using graphical method or using sign chart. The aim of this study is to provide alternative methods of finding the solution set of polynomial inequalities. In this research, the researcher presents general solutions to polynomial inequalities that can be linearly factored and all its real roots are with odd multiplicity and develops technique in solving polynomial inequality. The polynomial inequalities that considered are of degree $n \geq 3$ and roots of polynomial equation are of real roots with odd multiplicities. Thus, polynomial inequalities where $n \geq 3$ with even multiplicity and roots of the polynomial with non-zero imaginary part are not covered in this study. The techniques that is presented in this study will help engineers, physics and mathematicians in reducing the regorious steps of finding the solution set of a polynomial inequalities. Thus, using the results of the study, the solution to other mathematical problems that involves polynomial inequalities will be shortened. In 2015, Christopher
M. Cordero and Neil M. Mame of Batangas State University conducted the study “Algorithms in Solving Polynomial Inequality” and disclosed the solution set of polynomial inequalities with its root containing multiplicities. This study is related in a way that it is another method on finding the solution set of polynomial inequality that can be linearly factored and all the roots are with odd multiplicity. The researcher used this concept in proving the theorems of finding the solution set of a polynomial inequality where the roots of polynomial equation are real.

III. OBJECTIVES

The general objective of this study is to provide alternative methods of finding the solution set of polynomial inequalities of degree \( n > 2 \). Hence the specific objective of the study is to determine the solution set of a polynomial inequality with roots having odd multiplicity.

IV. METHODS

The methodology employed in this study is the analysis of end behavior of the polynomials, leading coefficient and degree of the polynomials and then from this information we can filter the solution set of the polynomial inequalities from a real line without taking any test point in between two consecutive roots. The study used the analysis of the graph of a polynomial functions and significant information needed in the study was gathered through research from the library and the World Wide Web. Since the purpose of the study is to introduce a new concept of solving polynomial inequalities, the researcher conducted a trial and error method to find a pattern in order to make a conjecture in certain cases, and then make a comparison with the results obtained by Christopher M. Cordero and Neil M. Mame to assure the generalization of the claim and illustration of a provided theorem.

V. SCOPE OF THE STUDY

The polynomial inequalities that considered are of degree \( n \geq 3 \) and roots of polynomial equation are of real roots with odd multiplicities. Thus, polynomial inequalities where \( n \geq 3 \) with even multiplicity and roots of the polynomial with non-zero imaginary part are not covered in this study.

VI. PRELIMINARY RESULTS

Fundamental Truths for Polynomial Functions All these statements are equivalent if \( c \in \mathbb{R} \). If one is true, all the others are true as well.
1) \( x = c \) is a root of the equation \( f(x) = 0 \).
2) \( c \) is a zero of the function \( f \).
3) \( c \) is an \( x \)-intercept of the graph of \( y = f(x) \).
4) \( x - c \) is a factor of \( f(x) \).

A. Solving polynomial inequalities using sign charts

A sign chart simply lists the sign of the function you are interested in along the \( x \)-axis. It does not provide as much information as a sketch, but it provides enough to solve inequalities.

Constructing a sign chart for a polynomial relies on knowing the end behaviour of the polynomial (which we get from the leading coefficient), and the \( x \)-intercepts.

1) Definition 1 (Multiplicity)

If the polynomial \( p_n(x) \) has \( (x - r)^m \) as a factor but not \( (x - r)^{m+1} \), then \( r \) is a zero of \( p_n(x) \) of multiplicity \( m \). If \( r \in \mathbb{R} \) is a zero of the polynomial \( p_n(x) \) with odd multiplicity, then the graph of \( p_n(x) \) crosses the \( x \)-axis at \( x = r \). i.e. The function \( p_n(x) \) changes sign at \( x = r \) and that is why the function \( p_n(x) \) changes its sign at \( x = r \).

2) End Behavior of a Polynomial

The end behavior of a polynomial function is a way to describe what is happening to the functional values as the \( x \)-values approach the ‘ends' of the \( x \)-axis. That is, what happens to \( y \) as \( x \) becomes small without bound (written \( x \to -\infty \)) and, on the flip side, as \( x \) becomes large without bound (written \( x \to \infty \)).

3) Leading Coefficient Test

Suppose \( p_n(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0 \), where \( a_0 \neq 0 \) is a real number. Then if \( n \) is an even natural number, the end behavior of the graph of \( y = p_n(x) \) matches one of the following:
   - for \( a_0 > 0 \), as \( x \to -\infty \), \( p_n(x) \to \infty \) and as \( x \to \infty \), \( p_n(x) \to \infty \)
   - for \( a_0 < 0 \), as \( x \to \infty \), \( p_n(x) \to -\infty \) and as \( x \to -\infty \), \( p_n(x) \to -\infty \)

If \( n \) is an odd natural number, the end behavior of the graph of \( y = p_n(x) \) matches one of the following:
   - for \( a_0 > 0 \), as \( x \to -\infty \), \( p_n(x) \to -\infty \) and as \( x \to \infty \), \( p_n(x) \to \infty \)
   - for \( a_0 < 0 \), as \( x \to -\infty \), \( p_n(x) \to \infty \) and as \( x \to \infty \), \( p_n(x) \to -\infty \)
4) The Intermediate Value Theorem
Suppose f is a continuous function on an interval containing \( x = a \) and \( x = b \) with \( a < b \). If \( f (a) \) and \( f (b) \) have different signs, then \( f \) has at least one zero between \( x = a \) and \( x = b \); that is, for at least one real number \( c \) such that \( a < c < b \), we have \( f (c) = 0 \).

B. Solving Polynomial Inequalities graphically using Multiplicity
Given \( p_n(x) = a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0 \) in standard form:
1) Write in completely factored form. (i.e. Find all the zeros and then write them as a product of linear factors.)
2) Plot real zeros on the x-axis, noting their multiplicity.
   - If the multiplicity is odd the function will go through the x-axis (\( p_n(x) \) changes its signs)
   - If the multiplicity is even the function will bounce off of the x-axis (\( p_n(x) \) do not change its sign)
3) Use the end-behavior to determine the sign of \( p_n(x) \) in the outermost intervals. Then work inward to label the other intervals as or by using the “change/no change” analysis of the multiplicity of neighboring zeros.
4) State the solution in interval notation.

Combining the property of multiplicity, end behavior and leading coefficient test together will leads us to formulate a solution set of a polynomial inequality on a real line without constructing a sign chart. This is the core concept that the researcher is going to phrase and attempt to put general solution set of polynomial inequalities of the form that is mentioned in the next section. Generally the researcher attempt to analyze the solution set of a polynomial inequality without taking any test point in between any two consecutive real roots of a polynomial. A number line containing the roots of a polynomial is written in their ascending order will be sketched and then we put alternating signs in between consecutive roots of a polynomial.

VII. Computing procedures

A. Case: 1
If \( r_i \in \mathbb{R} \) be the roots of a polynomial \( p_n = (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \) and assume that \( m_i \)’s are a positive odd integer. Let \( n \geq 3 \) be an even integer. Then we obtain the solution of the inequality \( (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} > 0 \) along the real line using the following steps:
   - Consider the end behavior of a polynomial of degree \( n \) if \( n \) is even positive integer with positive leading coefficient, as \( x \to -\infty \), \( p_n(x) \to \infty \) and as \( x \to \infty \), \( p_n(x) \to \infty \)
   - Assume the alternating sign change at all the roots since \( m_i \)’s are a positive odd integer.

Therefore, the solution set of the inequality can be read from the real line below without any computation.

Hence the solution set of \( (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} > 0 \) will be \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_2, r_3) \cup (r_4, r_5) \cup \ldots \cup (r_{2t-1}, r_{2t+1}) \cup \ldots \cup (r_k, \infty) \} \).

Thus the solution set of \( (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} \geq 0 \) is the set \( \{ x \in \mathbb{R} : x \in (-\infty, r_1] \cup [r_2, r_3] \cup [r_4, r_5] \cup \ldots \cup [r_{2t}, r_{2t+1}] \cup \ldots \} \).

Moreover, we can also tell about the solution set of the inequality \( (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} < 0 \) by observing the same chart that contains negatives intervals, that is \( \{ x \in \mathbb{R} : x \in (r_1, r_1+1) \} \) is odd and \( 1 \leq t \leq k \).

B. Case: 2
If \( r_i \in \mathbb{R} \) be the roots of a polynomial \( p_n = (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \) and assume that \( m_i \)’s are a positive odd integer. Let \( n \geq 3 \) be an even integer. Then we obtain the solution of the inequality \( (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} > 0 \) where \( c \) is any negative real number using the following steps:
   - Consider the end behavior of a polynomial of degree \( n \) if \( n \) is even positive integer with negative leading coefficient, as \( x \to -\infty \), \( p_n(x) \to \infty \) and as \( x \to \infty \), \( p_n(x) \to -\infty \)
   - Assume the alternating sign change at all the roots since \( m_i \)’s are a positive odd integer.

Therefore, the solution set of the inequality can be read from the real line below without any computation.

Hence the solution set of \( (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} > 0 \) will be \( \{ x \in \mathbb{R} : x \in (r_1, r_2) \cup (r_3, r_4) \cup (r_5, r_6) \cup \ldots \cup (r_{2t}, r_{2t+1}) \cup \ldots \cup (r_{2t-1}, r_1) \} \).

Thus the solution set of \( (x-r_1)^{m_1}(x-r_2)^{m_2}\ldots(x-r_k)^{m_k} \geq 0 \) is the set \( \{ x \in \mathbb{R} : x \in [r_1, r_2] \cup [r_3, r_4] \cup [r_5, r_6] \cup \ldots \cup [r_{2t}, r_{2t+1}] \cup \ldots \} \).

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Moreover we can also tell about the solution set of the inequality \( c((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k}) < 0 \) by observing the same chart that contains negative intervals, that is \( x \in \mathbb{R}: x \in (r_1, r_2) U (r_2, r_3) U (r_3, r_4) U \ldots U (r_n, r_{n+1}) U \ldots U (r_k, \infty) \), if \( t \) is even and \( 1 \leq t \leq k \).

**C. Case: 3**

If \( r_i \in \mathbb{R} \) be the roots of a polynomial \( p_n(x) = (r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k} \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \) and assume that \( m_i \)’s are a positive odd integer. Let \( n \geq 3 \) be an odd integer. Then we obtain the solution of the inequality \((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k} > 0 \) along the real line using the following steps:

- Consider the end behavior of a polynomial of degree \( n \) if \( n \) is odd positive integer with positive leading coefficient, as \( x \to -\infty, p_n(x) \to -\infty \) and as \( x \to \infty, p_n(x) \to \infty \)
- Assume the alternating sign change at all the roots since \( m_i \)’s are a positive odd integer.

Therefore, the solution set of the inequality can be read from the real line below without any computation.

![Fig. 3: Case 2](image.png)

Hence the solution set of \((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k} > 0 \) will be \( x \in \mathbb{R}: x \in (r_1, r_2) U (r_2, r_3) U (r_3, r_4) U \ldots U (r_n, r_{n+1}) U \ldots U (r_k, \infty) \), if \( t \) is odd and \( 1 \leq t \leq k \).

Thus the solution set of \((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k} \geq 0 \) is the set \( x \in \mathbb{R}: x \in [r_1, r_2] U [r_2, r_3] U [r_3, r_4] U \ldots U [r_{n-1}, r_n] U \ldots U [r_{k-1}, r_k] \), if \( t \) is odd and \( 1 \leq t \leq k \).

Moreover we can also tell about the solution set of the inequality \((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k} < 0 \) by observing the same chart that contains negative intervals, that is \( x \in \mathbb{R}: x \in (-\infty, r_1) U (r_2, r_3) U (r_4, r_5) U \ldots U (r_k, r_{k+1}) U \ldots U (r_n, \infty) \), if \( t \) is even and \( 1 \leq t \leq k \).

**D. Case: 4**

Let \( r_i \in \mathbb{R} \) be the roots of a polynomial \( p_n(x) = c((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k}) \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \) and assume that \( m_i \)’s are a positive odd integer. Let \( n \geq 3 \) be an odd integer. Then we obtain the solution of the inequality \( c((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k}) > 0 \) where \( c \) is a negative real number using the following steps:

- Consider the end behavior of a polynomial of degree \( n \) if \( n \) is odd positive integer with negative leading coefficient, as \( x \to -\infty, p_n(x) \to \infty \) and as \( x \to \infty, p_n(x) \to -\infty \)
- Assume the alternating sign change at all the roots since \( m_i \)’s are a positive odd integer.

Therefore, the solution set of the inequality can be read from the real line below without any computation.

![Fig. 4: Case 4](image.png)

Hence, the solution set of \( c((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k}) > 0 \) will be \( x \in \mathbb{R}: x \in (-\infty, r_1) U (r_2, r_3) U (r_4, r_5) U \ldots U (r_k, r_{k+1}) U \ldots U (r_n, \infty) \), if \( t \) is even and \( 1 \leq t \leq k \).

Thus, the solution set of \( c((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k}) \geq 0 \) is the set \( x \in \mathbb{R}: x \in (-\infty, r_1) U (r_2, r_3) U [r_4, r_5] U \ldots U [r_{n-1}, r_n] U \ldots U [r_{k-1}, r_k] \), if \( t \) is even and \( 1 \leq t \leq k \).

Moreover we can also tell about the solution set of the inequality \(-c((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k}) < 0 \) by observing the same chart that contains negative intervals, that is \( x \in \mathbb{R}: x \in (-\infty, r_1) U (r_2, r_3) U (r_4, r_5) U \ldots U (r_k, r_{k+1}) U \ldots U (r_n, \infty) \), if \( t \) is odd and \( 1 \leq t \leq k \).

- **Theorem 1:** Let \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( r_1 < r_2 < r_3 \). If \( p_n(x) < 0 \) for \( x \in (r_1, r_2) \) then the solution set is \( \{ x \in \mathbb{R}: x \in (r_1, r_2) \} \).
- **Corollary 1.1:** Let \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( r_1 < r_2 < r_3 \). If \( p_n(x) > 0 \) for \( x \in (r_1, r_2) \) then the solution set is \( \{ x \in \mathbb{R}: x \in [r_1, r_2] \} \).
- **Theorem 2:** Let \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( r_1 < r_2 < r_3 \). If \( p_n(x) > 0 \) for \( x \in (r_1, r_2) \) then the solution set is \( \{ x \in \mathbb{R}: x \in (-\infty, r_1) \} \).
- **Corollary 2.1:** Let \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( r_1 < r_2 < r_3 \). If \( p_n(x) < 0 \) for \( x \in (r_1, r_2) \) then the solution set is \( \{ x \in \mathbb{R}: x \in (-\infty, r_1) \} \).
- **Theorem 3:** Let \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) be the multiplicity of \( r_i \) and assume that \( m_i \)’s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). Let \( n \geq 3 \) be an even integer. Then \((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k} > 0 \) if and only if \( x \in \mathbb{R}: x \in (-\infty, r_1) U (r_1, r_1+i) \) or \( i \) is even.
- **Corollary 3.1:** Let \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) be the multiplicity of \( r_i \) and assume that \( m_i \)’s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). Then \((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k} \geq 0 \) if and only if \( x \in \mathbb{R}: x \in (-\infty, r_1) U [r_1, r_1+i] U [r_1, +\infty) \) or \( i \) is even.
- **Theorem 4:** Let \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) be the multiplicity of \( r_i \) and assume that \( m_i \)’s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). Let \( n \geq 3 \) be an odd integer. Then \((r_1)^{m_1}(r_2)^{m_2} \ldots (r_k)^{m_k} > 0 \) if and only if \( x \in \mathbb{R}: x \in (r_1, r_1+i) U (r_1, +\infty) \) or \( i \) is odd.
Corollary 4.1: Let \( r_1 \in \mathbb{R} \) be the roots of \( p_n \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). Let \( n \geq 3 \) an odd integer. Then \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \geq 0 \) if and only if \( \{ x \in \mathbb{R} : x \in [r_i, r_{i+1}) \cup [r_k, \infty) \ i \text{ is odd } \} \).

Theorem 5: Let \( r_i \in \mathbb{R} \) be the roots of \( p_n \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). If \( n \) is even then \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \leq 0 \) if and only if \( \{ x \in \mathbb{R} : x \in (r_i, r_{i+1}) \ i \text{ is odd } \} \).

Corollary 5.1: Let \( r_i \in \mathbb{R} \) be the roots of \( p_n \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). Let \( n \geq 3 \) an even integer. Then \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \leq 0 \) if and only if \( \{ x \in \mathbb{R} : x \in [r_i, r_{i+1}) \ i \text{ is odd } \} \).

Theorem 6: Let \( r_i \in \mathbb{R} \) be the roots of \( p_n \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). Let \( n \geq 3 \) an even integer. Then \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \leq 0 \) if and only if \( \{ x \in \mathbb{R} : x \in [r_i, r_{i+1}) \ i \text{ is even } \} \).

Corollary 6.1: Let \( r_i \in \mathbb{R} \) be the roots of \( p_n \) where \( 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). Let \( n \geq 3 \) an odd integer. Then \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \leq 0 \) if and only if \( \{ x \in \mathbb{R} : x \in (\infty, r_1) \cup [r_i, r_{i+1}) \ i \text{ is even } \} \).

VIII. FINDINGS

The findings of the study are as follows:
1. A polynomial inequality of degree three that can be factored as a product of linear factors with real roots.

   When \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( r_1 < r_2 < r_3 \). If \( (x-r_1)(x-r_2)(x-r_3) > 0 \), then there exists a solution set \( \{ x \in \mathbb{R} : x \in (r_1, r_2) \cup (r_2, \infty) \} \).

   Thus, if the inequality is \( (x-r_1)(x-r_2)(x-r_3) \geq 0 \), then the solution set is \( \{ x \in \mathbb{R} : x \in [r_1, r_2] \cup [r_2, \infty) \} \).

   When \( r_1, r_2 \) and \( r_3 \in \mathbb{R} \), where \( r_1 < r_2 < r_3 \). If \( (x-r_1)(x-r_2)(x-r_3) < 0 \), then the solution set is \( \{ x \in \mathbb{R} : x \in (\infty, r_1) \cup (r_2, \infty) \} \).

   Thus, if the inequality is \( (x-r_1)(x-r_2)(x-r_3) \leq 0 \), then the solution set is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_2, \infty) \} \).

2. Let \( (x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} > 0 \) where \( r_i \in \mathbb{R}, 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). If \( n \geq 3 \) an even integer, then the solution set is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_2, r_3), r_i \cup [r_i, r_{i+1}) \ i \text{ is even } \}. \)

   Thus, when \( (x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} \leq 0 \), then the solution set is \( \{ x \in \mathbb{R} : x \in [r_1, r_2) \cup [r_2, r_3) \cup [r_k, \infty) \ i \text{ is odd } \} \).

3. Let \( (x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} < 0 \) where \( r_i \in \mathbb{R}, 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). If \( n \geq 3 \) an odd integer, then the solution set is \( \{ x \in \mathbb{R} : x \in (r_1, r_2), r_i \cup (r_i, r_{i+1}) \ i \text{ is odd } \}. \)

4. Let \( (x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} \leq 0 \) where \( r_i \in \mathbb{R}, 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). If \( n \geq 3 \) an odd integer, then the solution set is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_2, r_3) \cup \ldots \cup [r_2, r_{k+1}) \cup \ldots \cup [r_k, \infty) \ i \in Z^+ \ i \leq 1 \leq k \}. \)

5. Let \( (x-r_1)^{m_1}(x-r_2)^{m_2}(x-r_3)^{m_3} > 0 \) where \( r_i \in \mathbb{R}, 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots < r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \). If \( n \geq 3 \) an odd integer, then the solution set is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_2, r_3) \cup \ldots \cup [r_2, r_{k+1}) \cup \ldots \cup [r_k, \infty) \ i \leq 1 \leq k \}. \)

IX. RECOMMENDATION

The results of the study can be used as an instructional reference material for instructors of mathematics. The finding can also help to introduce the students taking applied mathematics and other mathematics courses where the problems regarding polynomial inequality is required. The problem on solving polynomial inequalities whose roots are with even multiplicity is still open.

X. CONCLUSIONS

This study provided the general solution of a polynomial inequalities of degree greater than or equal to three where the roots of polynomial equation are real with odd multiplicity. When the roots of polynomial equation are real, then the solution set of \( p_0(x) = (x-r_1)(x-r_2)(x-r_3) > 0 \) is the set \( \{ x \in \mathbb{R} : x \in (r_1, r_2) \cup (r_3, \infty) \} \) thus when the inequality \( p_0(x) = (x-r_1)(x-r_2)(x-r_3) \geq 0 \) the
solution set is \( \{ x \in \mathbb{R} : x \in [r_1, r_2] \cup [r_3, \infty) \} \); if \( p_0(x) = (x-r_1)(x-r_2)(x-r_3) < 0 \) the solution set is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_2, r_3) \} \) thus, when the inequality \( p_0(x) = (x-r_1)(x-r_2)(x-r_3) \geq 0 \) the solution set is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1] \cup [r_2, r_3] \} \).

This paper also provided ways of finding the solution set of polynomial inequalities of degree \( n > 3 \) with real roots that have odd multiplicity. The first step is to let \( p_0(x) = (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \) and determine \( r_1, r_2, \ldots, r_k \) where \( r_1 < r_2 < \ldots, r_k \). Using end behavior of a polynomial and the alternating sign rule in between two consecutive real roots of a polynomial with odd multiplicity the inequality \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} > 0 \) where \( r_i \in \mathbb{R}, 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots, r_k \), \( m_i \) is the multiplicity of \( r_i \) and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \) if \( n \geq 3 \) an even integer, then the solution set is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_2, r_1+i) \cup (r_k, +\infty), i \text{ is even} \} \) by applying theorem 3 thus, when the inequality \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \geq 0 \) for \( n \) even the solution is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup [r_2, r_1+i] \cup [r_k, +\infty), i \text{ is even} \} \) by applying corollary 3.1; If \( n \) is odd or \( m_i = 1 \) for \( n \) odd then the solution set of \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \) is \( \{ x \in \mathbb{R} : x \in (r_2, r_1+i) \cup (r_k, +\infty), i \text{ is odd} \} \) by applying theorem 4 thus, when the inequality \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \geq 0 \) and \( n \) is odd then the solution is \( \{ x \in \mathbb{R} : x \in [r_1, r_1+i] \cup [r_k, +\infty), i \text{ is odd} \} \) by applying corollary 4.1.

Using similar argument on the end behavior of a polynomial and the oddness of the multiplicity the solution set of \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \leq 0 \) where \( r_i \in \mathbb{R}, 1 \leq i \leq k \) and \( r_1 < r_2 < \ldots, r_k \), \( m_i \) is the multiplicity of \( r_i \), and assume that \( m_i \)'s are a positive odd integer such that \( \sum_{i=1}^{k} m_i = n \) if \( n \geq 3 \) an even integer, then the solution set is \( \{ x \in \mathbb{R} : x \in (r_2, r_1+i) \cup (r_k, +\infty), i \text{ is odd} \} \) by applying theorem 5 thus, when the polynomial inequality \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \leq 0 \) by the corollary 5.1 when \( n \) is also even the solution set is \( \{ x \in \mathbb{R} : x \in [r_2, r_1+i] \cup [r_k, +\infty), i \text{ is even} \} \); if \( n \) is odd the solution set of \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \leq 0 \) is the set \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup (r_2, r_1+i), i \text{ is odd} \} \) by applying theorem 6 thus, when the polynomial inequality \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \leq 0 \) and \( n \) is odd or \( m_i \) is a positive odd integer then the solution set of \( (x-r_1)^{m_1}(x-r_2)^{m_2} \ldots (x-r_k)^{m_k} \) is \( \{ x \in \mathbb{R} : x \in (-\infty, r_1) \cup [r_2, r_1+i], i \text{ is even} \} \) by applying corollary 6.1.

Moreover the solution set of a polynomial inequality with negative leading coefficient will have the solution set of the form as is mentioned on the finding number 4 and 5.

References


[7] Dr. Claude Moore .(2012) MAT 171 Precalculus Algebra Cape Fear Community College


