Nadler Type Fixed Point Theorems for the Setting of Rhoades

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Abstract

Nadler has proved a fixed point theorem for XxZ where X is a complete metric space and Z is a metric space with the fixed point property. Rhoades discussed 125 contractive conditions and gave some results about these contractive conditions. In this paper we improve Nadler's results by weakening the conditions on the spaces and by enlarging the class of mappings by Rhoades type contractive conditions in the first variable. Here we extend the class of metric spaces Z by the class of uniform spaces for which Nadler type fixed point theorems holds.

Keywords: Complete Metric Space, Fixed Point Property, Locally Compact Space, Contractive Condition, Uniformly Continuous Mapping

I. INTRODUCTION

The main results of Nadler [5]:

A. Theorem 1.1

Let (X, d) be a metric space. Let A_i: X→X be a function with at least one fixed point a_i for each i=1,2,3,……,and let A_o :X→X be a contraction mapping with fixed point a_o. If the sequence $\{A_i\}$ converges uniformly to A_o, then the sequence $\{a_i\}$ converges to a_o.

B. Theorem 1.2

Let (X, d) be a locally compact metric space, let A_i: X→X be a contraction mapping with fixed point a_i for each i=1,2….If the sequence $\{A_i\}$ converges pointwise to A_o, then the sequence $\{a_i\}$ converges to a_o

C. Theorem 1.3

Let (X, d) be a complete metric space, Z a metric space which has the fixed point property (f.p.p.) and f: XxZ→XxZ be a contraction in the first variable.

- If f is uniformly continuous, then f has a fixed point.
- If (X, d) is locally compact, f is continuous, then f has a fixed point.

In what follows, X will denote a complete metric space, Z a uniform space in which sequences are adequate and f a mapping of XxZ into XxZ. For a fixed z∈Z, f_z: X→X be a mapping which is defined as $f_z(x)=\pi_1 f(x, z)$ for all x∈X, where π_1 is the projection of XxZ on X along Z. (m) for 1≤ m ≤ 125 will denote the condition (m) in Rhoades [6] with the modification that constants or functions that appear in (m) depend on z, Thus f_z ∈(23) means the following:

There exist monotonically decreasing functions α_i: (0,∞) →[0,1) (depend upon z) satisfying $\sum_{i=1}^{5} \alpha_i(t)<1$ such that for x, $x_* \in X$, $x \neq x_*$,

$$d(f_z(x), f_z(x_*)) \le \alpha_1(d(x, x_*)).d(x, f_z(x)) + \alpha_2(d(x, x_*)).d(x_*, f_z(x_*)) + \alpha_3(d(x, x_*)).d(x, f_z(x_*)) + \alpha_4(d(x, x_*)).d(x_*, f_z(x)) + \alpha_5(d(x, x_*)).d(x, x_*),$$

Where for brevity, we let $\alpha_i = \alpha_i(d(t_{n-1}, t_n))$.

Now we prove following results according to contractive conditions of Rhoades [6]:

D. Theorem 1.4

Let (X, d) be a complete metric space, Z a uniform space in which sequences are adequate and which has the f.p.p. and f: XxZ→XxZ be a mapping.

- If f is uniformly continuous and f_z ∈(23) for all z∈Z, then f has a fixed point.
- If (X, d) is locally compact, f is continuous and f_z ∈(23) for all z∈Z, then f has a fixed point.

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1) Proof

We prove (a) and (b) parts simultaneously:

a) Step 1: $\{t_n\}$ is a Cauchy sequence in X

For $z \in Z$, we define the sequence $t_n(z) = t_n$ in X as follows:

For a fixed x_0 in X, $t_0 = x_0$, $t_n = \pi_1 f(t_{n-1}, z) = f_z^n(t_0)$; $n \geq 1$.

Since $f_z \in (23)$, then we have (taking $x = t_{n-1}$ and $x_* = t_n$):

$$d(f_z(t_{n-1}), f_z(t_n)) \leq \alpha_1 d(t_{n-1}, f_z(t_{n-1})) + \alpha_2 d(t_n, f_z(t_n)) + \alpha_3 d(t_{n-1}, f_z(t_n)) + \alpha_4 d(t_n, f_z(t_{n-1})) + \alpha_5 d(t_{n-1}, t_n)$$

or

$$d(t_n, t_{n+1}) \leq \alpha_1 . d(t_{n-1}, t_n) + \alpha_2 d(t_n, t_{n+1}) + \alpha_3 d(t_{n-1}, t_{n+1}) + \alpha_5 d(t_{n-1}, t_n), \quad (1)$$

Using symmetry in (23) we have

$$d(t_{n+1}, t_n) \leq \alpha_1 d(t_n, t_{n+1}) + \alpha_2 d(t_{n-1}, t_n) + \alpha_4 d(t_{n-1}, t_{n+1}) + \alpha_5 d(t_n, t_{n-1}) \quad (2)$$

Adding equation (1) and (2) we obtain

$$\begin{aligned} 2d(t_n, t_{n+1}) &\leq (\alpha_1 + \alpha_2 + 2\alpha_5) d(t_{n-1}, t_n) + (\alpha_1 + \alpha_2) d(t_n, t_{n+1}) + (\alpha_3 + \alpha_4) . d(t_{n-1}, t_{n+1}) \\ &\leq (\alpha_1 + \alpha_2 + 2\alpha_5) . d(t_{n-1}, t_n) + (\alpha_1 + \alpha_2) . d(t_n, t_{n+1}) + (\alpha_3 + \alpha_4) [d(t_{n-1}, t_n) + d(t_n, t_{n+1})] \\ &= (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5) . d(t_{n-1}, t_n) + (\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) . d(t_n, t_{n+1}) \end{aligned}$$

Or

$$d(t_n, t_{n+1}) \leq \left(\frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5}{2 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4} \right) d(t_{n-1}, t_n) = \lambda \, d(t_{n-1}, t_n), \quad \ldots\ldots(3)$$

Where

$$\left(\frac{\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + 2\alpha_5}{2 - \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4} \right) = \lambda < 1 \text{ as } \sum_{i=1}^{5} \alpha_I < 1.$$

By induction we obtain $d(t_n, t_{n+1}) \leq \lambda^n d(t_0, t_1)$.

Using triangle inequality, we find for $m > n$

$$\begin{aligned} d(t_n, t_m) &\leq d(t_n, t_{n+1}) + d(t_{n+1}, t_{n+2}) + \ldots + d(t_{m-1}, t_m) \\ &\leq (\lambda^n + \lambda^{n+1} + \ldots\ldots + \lambda^{m-1}) d(t_0, t_1) \\ &= \frac{\lambda^n (1 - \lambda^{m-n})}{1 - \lambda} d(t_0, t_1) < \left(\frac{\lambda^n}{1 - \lambda} \right) d(t_0, t_1) \end{aligned}$$

Since $\lambda^n \to 0$ as $n \to \infty$, then above inequality shows that $\{t_n\}$ is a Cauchy sequence in X.

Since X is complete metric space, there is a point p_1 in X such that $\lim t_n = p_1$.

b) Step 2: We show that p_1 is a unique fixed point of f_z:

We shall first show that sequence $\{t_{n+1}\}$ converges to $f_z(p_1)$. Since $f_z \in (23)$ and if we let $x = t_n$, $x_* = p_1$.

Then assuming that $t_n \neq p_1$. We have,

$$\begin{aligned} d(t_{n+1}, f_z(p_1)) &\leq \alpha_1 d(t_n, t_{n+1}) + \alpha_2 d(p_1, f_z(p_1)) + \alpha_3 d(t_n, f_z(p_1)) + \alpha_4 d(p_1, t_{n+1})) + \alpha_5 d(t_n, p_1). \\ &\leq \alpha_1 d(t_n, t_{n+1}) + \alpha_2 [d(p_1, t_{n+1}) + d(t_{n+1}, f_z(p_1))] + \alpha_3 [d(t_n, t_{n+1}) + d(t_{n+1}, f_z(p_1))] + \alpha_4 d(p_1, t_{n+1}) + \alpha_5 d(t_n, p_1). \end{aligned}$$

Or

$$d(t_{n+1}, f_z(p_1)) \leq \frac{(\alpha_1 + \alpha_2) . d(t_n, t_{n+1}) + (\alpha_2 + \alpha_4) . d(p_1, t_{n+1}) + \alpha_5 d(t_n, p_1)}{1 - \alpha_2 - \alpha_3} \quad \ldots\ldots(4)$$

Similarly, if we let $x = t_n$, $x_* = p_1$ and use symmetry for $f_z \in (23)$ then we have,

$$d(f_z(p_1), t_{n+1}) \leq \frac{(\alpha_1 + \alpha_3) . d(p_1, t_{n+1}) + (\alpha_2 + \alpha_4) . d(t_n, t_{n+1}) + \alpha_5 d(p_1, t_n)}{1 - \alpha_1 - \alpha_4} \quad \ldots\ldots(5)$$

Since $\alpha_2 + \alpha_3$ and $\alpha_1 + \alpha_4$ are both evaluated at $d(t_n, p_1)$ and since $\sum_{i=1}^{5} \alpha_i(t) < 1$ for each $t = d(t_n, p_1) > 0$ it follows that at least one of the sums, say $\alpha_2 + \alpha_3$ must be less than ½ for an infinite number of choices n_i of n, therefore both equations (4) and (5) implies that $d(f_z(p_1), t_{ni+1}) = 0$. Since $d(t_n, t_{n+1})$ is monotonic decreasing in n, we can conclude that sequence $\{t_{n+1}\}$ converges to $f_z(p_1)$. Hence $p_1 = f_z(p_1)$ or p_1 is a fixed point of f_z.

Uniqueness of fixed point:

Suppose p_2 be another fixed point of f_z such that $p_1 \neq p_2$, then we have

$$\begin{aligned} d(p_1, p_2) &= d(f_z(p_1), f_z(p_2)) \\ &\leq \alpha_1 d(p_1, f_z(p_1)) + \alpha_2 d(p_2, f_z(p_2)) + \alpha_3 d(p_1, f_z(p_2)) + \alpha_4 d(p_2, f_z(p_1)) + \alpha_5 d(p_1, p_2) \\ &= (\alpha_3 + \alpha_4 + \alpha_5) . d(p_1, p_2) \end{aligned}$$

Which is contradiction as $d(p_1, p_2) \neq 0$ and $\sum_{i=1}^{5} \alpha_i(t) < 1$. Hence $p_1 = p_2$.

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c) Step 3: Let a mapping F: Z→X be such that F(z) = p_1 where p_1 is the unique fixed point of f_z.

Now, let $z_0 \in Z$ and let $\{z_i\}$ is a sequence in Z which converges to z_0. Then by the hypothesis (a), the sequence $\{f_{zi}\}$ converges uniformly to f_{z0} and hence by the Theorem 1.1, the sequence $\{F(z_i)\}$ converges to $F(z_0)$. Under the assumptions of (b), we may apply the Theorem 1.2 conclude that the sequence $\{F(z_i)\}$ converges to $F(z_0)$. Hence in either case, this proves that F is continuous on Z. Also $\pi_1 f(F(z),z)=f_z(F(z))=F(z)$, because F(z) is a fixed point of f_z. Next, let G: Z→Z be defined by setting $G(z)= \pi_2 f(F(z),z)$, where π_2 is the projection of XxZ on Z along X. Then G is a continuous map of Z to itself. Since Z has the f.p.p., there exists a point $p \in Z$ such that G(p) = p. Then the point (F(p), p) is such that $\pi_1 f(F(p), p) = F(p)$ and $\pi_2 f(F(p), p) = G(p) = p$, therefore (F(p), p) is a fixed point of f in XxZ.

Rhoades [6] proved that conditions (1), (2), (4), (7), (8), (11), (15) and (18) are all stronger than (23), therefore the above Theorem 1.4 has eight corollaries corresponding to each of these eight conditions.

Our next result is corresponding to Theorem 3 of Rhoades [6]. We prove (a) and (b) parts simultaneously.

E. Theorem 1.5

Let (X, d) be a complete metric space, Z a uniform space in which sequences are adequate and which has the f.p.p. Let f: XxZ→XxZ be a mapping.

- If f is uniformly continuous and $f_z \in$(22) for all $z \in Z$ and that for some $x_0 \in X$, the sequence, $t_0 = x_0$, $t_n(z) = t_n = \pi_1 f(t_{n-1}, z) \equiv f_z^n(t_0)$; $n \geq 1$ has a cluster point, then f has a fixed point.
- If (X, d) is locally compact, f is continuous and $f_z \in$(22) for all $z \in Z$ and for some $x_0 \in X$, the sequence $\{t_n\}$defined in (a) has a cluster point, then f has a fixed point.

1) Proof: Step 1: $\{t_n\}$ is a Cauchy sequence in X:

Since $f_z \in$(22) and if we let $x = t_{n-1}$, $x_* = t_n$ then we have $d(f_z(t_{n-1}), f_z(t_n)) < \max \{d(t_{n-1}, t_n), d(t_{n-1}, f_z(t_{n-1})), d(t_n, f_z(t_n)), [d(t_{n-1}, f_z(t_n)) + d(t_n, f_z(t_{n-1}))]/2\}$

or

$$d(t_n, t_{n+1}) < \max\{d(t_{n-1}, t_n), d(t_n, t_{n+1}), [d(t_{n-1}, t_{n+1}]/2\}$$
$$\leq \max \{d(t_{n-1}, t_n), d(t_n, t_{n+1}), [d(t_{n-1}, t_n) + d(t_n, t_{n+1})]/2\}$$

Or

$$d(t_n, t_{n+1}) < d(t_{n-1}, t_n) \leq \lambda\, d(t_{n-1}, t_n) \text{ where } \lambda \in [0, 1)$$

By induction we obtain $d(t_n, t_{n+1}) \leq \lambda^n d(t_0, t_1)$

Using triangle inequality, we find for m > n;

$$d(t_n, t_m) = d(f_z^n(t_0), f_z^m(t_0)) = d(f_z^n(t_0), f_z^n f_z^{m-n}(t_0))$$
$$\leq \lambda^n d(t_0, f_z^{m-n}(t_0))$$
$$= \lambda^n d(t_0, t_{m-n})$$
$$\leq \lambda^n [d(t_0, t_1) + d(t_1, t_2) + \ldots\ldots\ldots\ldots + d(t_{m-n-1}, t_{m-n})]$$
$$\leq \lambda^n d(t_0, t_1)[1 + \lambda + \ldots\ldots\ldots + \lambda^{m-n-1}]$$
$$< \frac{\lambda^n}{1-\lambda} d(t_0, t_1)$$

Since $\lambda^n \to 0$ as $n \to \infty$, the above inequality shows that $\{t_n\}$ is a Cauchy sequence in X. Since X is complete, there is a point p_1 in X such that $t_n \to p_1$.

2) Step 2:

We show that p_1 is a unique fixed point of f_z:

Since $f_z \in$(22). Let $x = t_n$, $x_* = p_1$ and assuming that $t_n \neq p_1$with $\lim t_n = p_1$. Then we have.

$$d(f_z(t_n), f_z(p_1)) < \max \{d(t_n, p_1), d(t_n, f_z(t_n)), d(p_1, f_z(p_1)), [d(t_n, f_z(p_1)) + d(p_1, f_z(t_n))]/2\}$$

or

$$d(t_{n+1}, f_z(p_1)) < \max\{d(t_n, p_1), d(t_n, t_{n+1}), d(p_1, f_z(p_1)), [d(t_n, f_z(p_1)) + d(p_1, t_{n+1})]/2\}$$

Taking $n \to \infty$ we have

$$d(p_1, f_z(p_1)) < \max\{d(p_1, p_1), d(p_1, f_z(p_1)), [d(p_1, f_z(p_1)) + d(p_1, p_1)]/2\}$$

or

$$d(p_1, f_z(p_1)) < d(p_1, f_z(p_1)),$$

Which is contradiction. Hence $p_1 = f_z(p_1)$ i.e. p_1 is a fixed point of f_z.

For the uniqueness of a fixed point suppose p_2 be another fixed point of f_z such that $p_1 \neq p_2$ then we have from $f_z \in$(22), $d(p_1, p_2) = d(f_z(p_1), f_z(p_2)) < \max\{d(p_1, p_2), d(p_1, f_z(p_1)), d(p_2, f_z(p_2)), [d(p_1, f_z(p_2)) + d(p_2, f_z(p_1))]/2\}$.

Which implies that $d(p_1, p_2) < d(p_1, p_2)$.

Which is contradiction to $p_1 \neq p_2$. Hence $p_1 = p_2$.

By the help of step III of the above theorem 1.4, we conclude that (f(p), p) is a fixed point of f.

In view of Theorem.1 of Rhoades [6], the Theorem 1.5 above has as many as eleven corollaries corresponding to conditions (1), (4), (5), (6), (7), (9), (11), (18), (19), (20) and (21). The corollaries corresponding to conditions (1), (4), (7), (11) and (18) are

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also corollaries to theorem 1.4, Which are mentioned above. Therefore, remaining corollaries are corresponding to conditions (5), (6), (9), (19) (20) and (21).

In part (b) of the above Theorem we cannot drop the continuity of f as is shown by the following example:

a) Example (A)

Let $X=\{x_n = 1-2^{-n-1}: n\geq 0\} \cup \{1\}$ and $Z=\{z\}$ be a single point uniform space. We define $f: XxZ \to XxZ$ by setting $f(1, z) = (1/2, z)$ and $f(x_n, z) = (x_{n+1}, z)$. Then X is locally compact and f is not continuous. Also $f_z \in$(22). In fact $f_z \in$ (6) and we know that condition (6) implies condition (22). If we take $x_o=1 \in X$, then the sequence t_n defined in (a) is as follows:

$$t_o = x_o = 1,\ t_1 = \pi_1 f(1, z)=1/2 \text{ and } t_n = \pi_1 f(t_{n-1}, z)= x_{n+1} \text{ for } n \geq 2.$$

Hence the sequence $\{t_n\}$ has a cluster point. Obviously the function $f: XxZ \to XxZ$ has no fixed point.

Our next result is corresponding to Theoem.5 of Rhoades [6]. We prove (a) and (b) parts simultaneously.

F. Theorem 1.6

Let (X, d) be a complete metric space, Z a uniform space in which sequences are adequate which has the f.p.p. and $f:XxZ \to XxZ$ be a mapping.

- If f is uniformly continuous and $f_z \in$(24) for all $z \in Z$, then f has a fixed point.
- If (X,d) is locally compact, f is continuous and $f_z \in$(24) for all $z \in Z$, then f has a fixed point.

1) Proof: Step 1: $\{t_n\}$ is a Cauchy sequence in X:

For each $z \in Z$, we define the sequence $t_n(z) = t_n$ in X as follows:

For a fixed x_o in X, $t_o= x_o$, $t_n= \pi_1 f(t_{n-1}, z) \equiv f_z^n(t_0)$; $n \geq 1$

Since $f_z \in$(24) and if we let $x = t_{n-1}$, $x_*= t_n$ then we have,

$$d(f_z(t_{n-1}), f_z(t_n)) \leq h.\ \max \{d(t_{n-1}, t_n), d(t_{n-1}, f_z(t_{n-1})), d(t_n, f_z(t_n)), d(t_{n-1}, f_z(t_n)), d(t_n, f_z(t_{n-1}))\}$$

or

$$d(t_n, t_{n+1}) \leq h.\ \max\{d(t_{n-1}, t_n), d(t_n, t_{n+1}), d(t_{n-1}, t_{n+1})\}.$$

Which implies that $d(t_n, t_{n+1}) \leq h.\ d(t_{n-1}, t_n)$.

By induction we have $d(t_n, t_{n+1}) \leq h^n d(t_0, t_1)$

Using triangle inequality, we find for $m > n$;

$$d(t_n, t_m) = d(f_z^n(t_0), f_z^m(t_0)) = d(f_z^n(t_0), f_z^n f_z^{m-n}(t_0))$$

$$\leq h^n d(t_0, f_z^{m-n}(t_0)) = h^n d(t_0, t_{m-n})$$

$$\leq h^n [d(t_0, t_1)+d(t_1, t_2)+\text{---}+d(t_{m-n-1}, t_{m-n})]$$

$$\leq h^n d(t_0, t_1)[1+h+\text{----}+h^{m-n-1}]$$

$$\leq \frac{h^n}{1-h} d(t_o, t_1)$$

Since $h^n \to 0$ as $n \to \infty$, the above inequality shows that $\{t_n\}$ is a Cauchy sequence in X.

Since X is complete metric space, therefore a point p_1 in X such that t_n converges to p_1.

2) Step 2: We show that p_1 is a unique fixed point of f_z :

Since $f_z \in$(24). Let $x = t_n$, $x_* = p_1$ and assuming that $t_n \neq p_1$ with $\lim t_n = p_1$. Then we have.

$$d(f_z(t_n), f_z(p_1)) \leq h.\max \{d(t_n, p_1), d(t_n, f_z(t_n)), d(p_1, f_z(p_1)), d(t_n, f_z(p_1)), d(p_1, f_z(t_n))\}.$$

Or

$$d(t_{n+1}, f_z(p_1)) \leq h.\max\{d(t_n, p_1), d(t_n, t_{n+1}), d(p_1, f_z(p_1)), d(t_n, f_z(p_1)), d(p_1, t_{n+1})\}$$

Taking $n \to \infty$ then we have $d(p_1, f_z(p_1)) \leq h.d(p_1, f_z(p_1))$.

Which is contradiction, hence $f_z(p_1) = p_1$ or p_1 is a fixed point of f_z.

Uniqueness of fixed point is as follows:

If possible let p_2 be another fixed point of f_z such that $p_1 \neq p_2$, then we have

$d(p_1, p_2) = d(f_z(p_1), f_z(p_2)) \leq h.\max\{d(p_1, p_2), d(p_1, f_z(p_1)), d(p_2, f_z(p_2)), d(p_1, f_z(p_2)), d(p_2, f_z(p_1))\}$ or $d(p_1, p_2) \leq h.d(p_1, p_2) < d(p_1, p_2)$ which is a contradiction. Hence $p_1 = p_2$.

By the help of step III of theorem 1.4 we have (F(p), p) is a fixed point of f.

In view of Theorem 1 of Rhoades [6], our Theorem 1.6 above has twelve corollaries corresponding to conditions (1), (4), (5), (7), (9), (11), (12), (14), (16), (18), (19) and (21), and out of these only three are new corollaries (corresponding to (12), (14) and (16)).

Rhoades [6] has also shown that (m) $\Rightarrow$(25+m)$\Rightarrow$(50+m); $1 \leq m \leq 25$ and that for condition (72), (73) and (74) fixed point theorems are available. Making suitable hypothesis the mapping satisfying these conditions have unique fixed points. As we have seen, this uniqueness of fixed point is sufficient to define the mapping $F: Z \to X$. Therefore, by using suitable conditions the three theorems can be generalized to mappings f such that for all $z \in Z$, $f_z \in$(72) or (73) or (74).

Our next results are corresponding to conditions (72), (73) and (74) which are generalization of the Theorems 1.5, 1.4 and 1.6 proofs of these results are obtained with the help of Theorems 9, 10 and 11 of Rhoades [6]. These results are formulated as follows:

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G. Theorem 1.7

Let (X, d) be a complete metric space, Z a uniform space in which sequences are adequate which has the f.p.p. and f:XxZ→XxZ be a mapping.

- If f is uniformly continuous and for each z∈Z there exist positive integer p and q such that for each x, $x_* \in X$, $x \neq x_*$, $d(f_z^p(x), f_z^q(x_*)) < \max\{d(x, x_*), d(x, f_z^p(x)), d(x_*, f_z^q(x_*)), \frac{[d(x_*, f_z^p(x)) + d(x, f_z^q(x_*))]}{2}\}$, then f has a fixed point.
- If X is locally compact, f is continuous and for each z∈Z, f_z satisfies the condition in (a), then f has a fixed point.

H. Theorem 1.8

Let (X, d) be a complete metric space, Z a uniform space in which sequences are adequate and which has the f.p.p. Let f: XxZ→XxZ is a mapping.

- If f is uniformly continuous and for each z∈Z there exist monotonic decreasing functions $\propto_i : (0,\infty) \to [0,1)$ satisfying $\sum \alpha_i(t) < 1$ (for each i=1,2,…5) and positive integers p and q such that for each x,$x_* \in X$, $x \neq x_*$, $d(f_z^p(x), f_z^q(x_*)) \leq \alpha_1(d(x, x_*))\, d(x, f_z^p(x)) + \alpha_2(d(x, x_*))\, d(x_*, f_z^q(x_*)) + \alpha_3(d(x, x_*))\, d(x, f_z^q(x_*)) + \alpha_4(d(x, x_*)).d(x_*, f_z^p(x)) + \alpha_5(d(x, x_*)).d(x, x_*)$, then f has a point.
- If X is locally compact, f is continuous and for each z∈Z, f_z satisfies the condition in (a), then f has a fixed point.

I. Theorem 1.9

Let (X, d) be a complete metric space, Z a uniform space in which sequences are adequate which has the f.p.p. and f: XxZ→XxZ be a mapping.

- If f is uniformly continuous and for each z∈Z, there exist a real number h∈[0,1) and positive integers p and q such that for each x, $x_* \in X$, $d(f_z^p(x), f_z^q(x_*)) \leq h.\max\{d(x, x_*), d(x, f_z^p(x)), d(x_*, f_z^q(x_*)), d(x, f_z^q(x_*)), d(x_*, f_z^p(x))\}$, then f has a fixed point.
- If X is locally compact, f is continuous and for each z∈Z, f_z satisfies the condition in (a), then f has a fixed point.

Here we note that, for the proofs of above Theorems, $f_z^p(x)$ stands for $\pi_1 f(f_z^{p-1}(x), z)$ and $f_z^0(x_0) = t_0$. The sequence $\{t_n\}$ is constructed as descried above and $\{t_n\}$converges to a unique point of X irrespective of the choice of the initial point t_o.

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