A Steady State Heat Transfer Analysis of Semi Circular Plate with Prescribed Circular Boundary Condition by using Laplace's Equation

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Abstract

In this paper we study the heat transfer analysis in a homogeneous lamina with circular boundaries under steady state conditions. This study becomes more important while designing various control panels and control plates in a test rig. The governing Laplace's equations in (r,θ) Coordinates for a two dimensional case is considered. The solution of Laplace's equation results in an analytical solution. The same heat transfer analysis is carried out using ANSYS 12.0 for validation. We have found that the results are satisfactory. The details of results, analysis and discussions are presented.
Keywords: (x,y,z)= Spatial co-ordinates along x, y and z axes , m, r. (r,θ) = polar coordinates m, degrees (r,θ,z)= Axes, m, degrees, m t-Time co-ordinate, sec u-temperature(degree C) T- Temperature (degree C)

I. INTRODUCTION

Many problems in engineering exists which are either Laplace's equations or Poisson's equations. But currently for our study we have chosen a problem of heat transfer in a circular lamina with prescribed boundary conditions. The study as well as the results is important in the design of control plates in control systems. The governing equations are two dimensional Laplace's equations in two dimensions (r, θ) system. The solution is obtained by the method of separation of variables. The results and analysis are discussed.

Mathematics in the problem under study:

To study heat flow equations in steady state, we consider

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \ (1) \text{ in two dimensions}$$

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \ \ldots\ldots\ldots\ldots \ (2) \text{ in three dimensions}$$

These equations have made applications in physics and engineering. The theory of potential gives solutions which are called harmonic functions. The solution of Laplace's equation, subject to certain boundary conditions is simplified by a proper selection of coordinate system.

If a problem involves rectangular boundaries, we prefer to take Laplace's equation in Cartesian coordinates given by equations (1) & (2).

If the problem involves circular boundaries, we take Laplace's equation in polar coordinates given by

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \ \ldots\ldots\ldots\ldots \ (3)$$

This equation can be obtained from equation

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

By putting $x = r\cos\theta$, $y = r\sin\theta$ effectively, changing the independent variables from (x, y) to (r, θ).

If the problem involves cylindrical boundaries, it is preferred to take Laplace's equation in cylindrical coordinates given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{\partial^2 u}{\partial z^2}=0\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

In this case, this equation can be obtained by substituting x =rcosθ , y=rsinθ , z=z. Thus changing the independent variables (x,y,z) to (r,θ,z).

The problems with spherical boundaries, we prefer to take Laplace's equation in spherical coordinates given by

$$\frac{\partial^2 u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}+\frac{cot\theta}{r}\frac{\partial u}{\partial \theta}+\frac{1}{r^2}\frac{1}{Sin^2\theta}\frac{\partial^2 u}{\partial \theta^2}$$

This equation can be obtained by putting $x = rsin\theta cos\emptyset$ and $y = rsin\theta sin\emptyset$, $z = rcos\theta$ thus changing independent variables (x, y, z) to $(r, \theta, \emptyset)$.

The solution of Laplace's equation in polar coordinates: Laplace's equation in polar coordinates is,

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

By the method of separation of variables, we assume the solution as
$u(r, \theta) = R(r).F(\theta)$ or simply

$$u = RF \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6)$$

Where R is a function of r only and F is a function θ only, be solution of (5)
Substituting in (5) we get
$r^2R''F + rR'F + RF'' = 0$ or
$F(r^2R'' + rR') + RF'' = 0$
Separating the variables, we get

$$\frac{r^2R''+rR'}{R}=\frac{F''}{F}=a\ constant=k(say)$$

Thus we get ordinary differential equation,

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-kR=0\ldots\ldots\ldots\ldots\ldots\ldots (7)$$

$$\frac{d^2F}{d\theta^2}+kF=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8)$$

Now equation (7) is s homogeneous linear differential equation.
Putting R = e^Z equation (7) reduces to

$$\frac{d^2R}{dz^2}-kR=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (9)$$

$$\frac{d^2F}{d\theta^2}+kF=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Case (i) When k is a positive
Substituting $k = P^2$, we get

$$R = c_1r^p + c_2r^{-p}$$

Similarly, for equation

$$\frac{d^2F}{d\theta^2}+kF=0\text{ , we get,}$$

$$F = c_3cosp\theta + c_4sinp\theta$$

Case (ii) When k is negative and $-P^2$ (say), Substituting r = $-P^2$ =>

$$R = c_1cosp(logr) + c_2sinp(logr)$$

$$F = c_3e^{p^\theta} + c_4e^{-p^\theta}$$

Case (iii) when k = 0

$$R = c_1z + c_2 = c_1logr + c_2$$

$$F = c_3\theta + c_4$$

Thus the three possible solutions of (1) are,

$$U = (c_1r^p + c_2r^{-p})(c_3cosp\theta + c_4sinp\theta)\ \ldots\ldots\ldots\ldots\ldots\ldots (11)$$

$$U = \left(c_1cosp(logr) + c_2sinp(logr)\right)(c_3e^{p\theta} + c_4e^{-p\theta})\ \ldots\ldots\ldots (12)$$

$$U = (c_1logr + c_2)(c_3\theta + c_4))\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Of these solutions, we choose the one which is consistent with the physical nature of the problem. Usually we require a solution up to the origin. Since u must be finite at the origin, we reject the solution (12) & (13. Also from (11) $c_2 = 0$
Hence, $u = (Acosp\theta + Bsinp\theta)^{r^p}$
The lamina under study is a semicircular body with centre of the circle as the pole and bounding diameter as the initial line. Let the steady temperature at $\theta)$ any point P(r, θ) be
$u(r, \theta)$ so that u satisfies the equation

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}=0\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (14)$$

The boundary conditions are

$$u(r,0) = 0 \quad in \quad 0 \le r \le a \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(15)}$$
$$u(r,0) = 0 \quad in \quad 0 \le r \le a \ldots\ldots\ldots\ldots\ldots\ldots \text{(16)}$$
$$u(a,\theta) = T \quad \ldots\ldots\ldots\ldots\ldots\ldots \text{(17)}$$

Solution:

$$u(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_3 cosp\theta + c_4 sinp\theta) \ldots\ldots\ldots \text{(18)}$$

Applying condition (15)

$$u(r,\theta) = (c_1 r^p + c_2 r^{-p})c_3 = 0$$

Therefore $c_3 = 0$ And equation (18) becomes

$$u(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_4 sinp\theta) \ldots\ldots\ldots \text{(19)}$$

Applying condition 16

$$u(r,\theta) = (c_1 r^p + c_2 r^{-p})(c_4 sinp\pi) = 0$$

Or $sin\pi = 0$, hence $p = n$ where n is an integer

Therefore equation (19) reduces to

$$u(r,\theta) = (c_1 r^n + c_2 r^{-n})(c_4 sinn\theta) = 0 \ldots\ldots\ldots \text{(20)}$$

Since $u = 0$ when $r = 0$, hence $c_2 = 0$

And equation (20) becomes

$u(r,\theta) = b_n r^n sinn\theta$ where $b_n = c_1 c_4$

so, the most general solution of (1) is of the form

$$u(r,\theta) = \textstyle\sum_{n=1}^{\infty} b_n\, r^n sinn\theta \quad \ldots\ldots\ldots \text{(21)}$$

applying condition (4)

$$u(a,\theta) = \sum_{n=1}^{\infty} b_n\, r^n sinn\theta = T$$

Or $T = \sum_{n=1}^{\infty} b_n\, r^n sinn\theta$ where $B_n = b_n a^n$

$$B_n = \frac{2}{\pi}\int_0^{\pi} Tsinn\theta d\theta = \frac{2T}{nm}(1 - cosn\pi)$$

$$b_n = \frac{B_m}{a^n} = \frac{2T}{nma^n}(1 - cosn\pi) = \begin{cases} 0 \;\; if\ n\ is\ even \\ \dfrac{2T}{n\pi a^n} \;\; if\ n\ is\ odd \end{cases}$$

Hence from (21) we have

$$u(r,\theta) = \frac{4T}{\pi}[\frac{\left(\frac{r}{a}\right)}{1} sin\theta + \frac{\left(\frac{r}{a}\right)^3}{3} sin3\theta + \frac{\left(\frac{r}{a}\right)^5}{5} sin5\theta + \cdots \ldots \ldots]$$

$$u(r,\theta) = \frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{(2n-1)}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)^{\theta}$$

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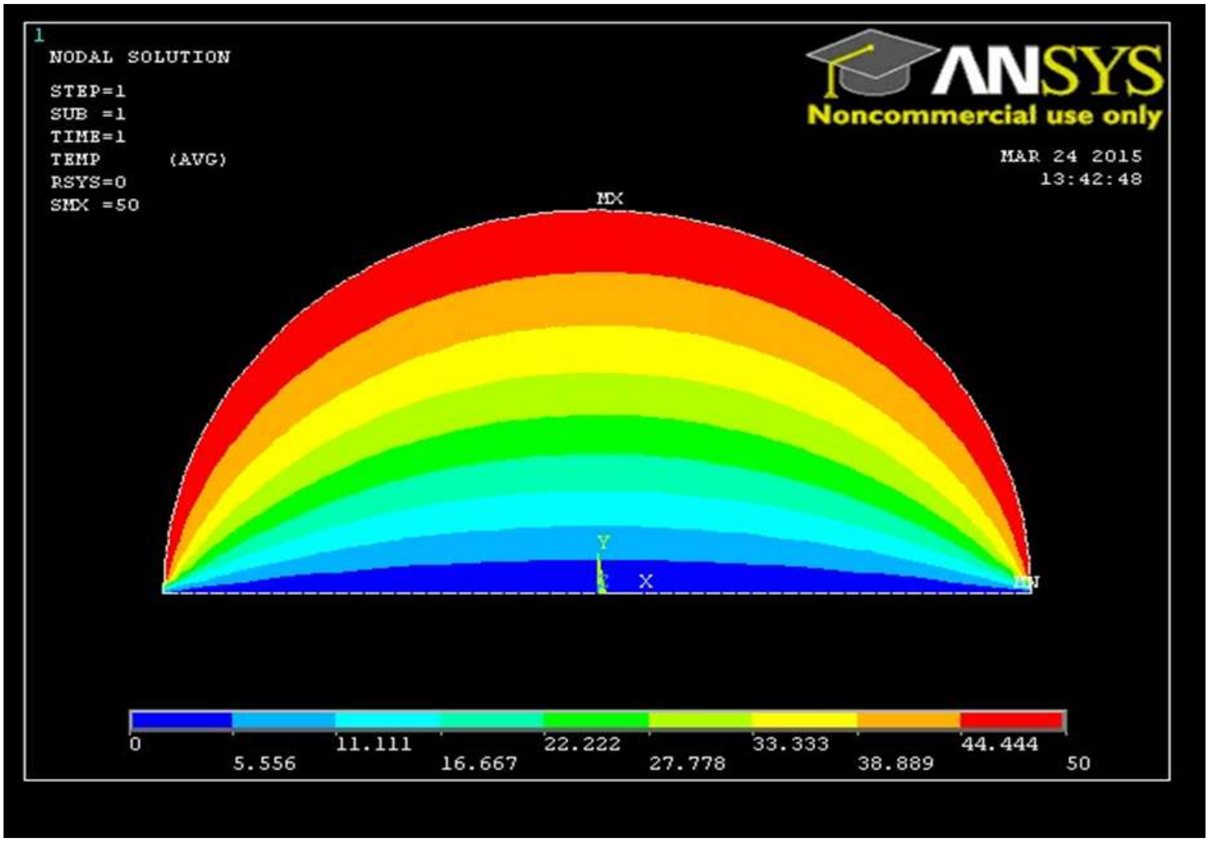
The heat transfer analysis through the semicircular slab was also evaluated by using ANSYS software version 12. 0. The important input chosen for the analysis include i) Element type solid 4 node 55

1) Material properties; Thermal conductivity K of the material
2) Boundary temperatures.

The analysis was carried out for steady state heat transfer and the obtained results are presented in the form of contour plots.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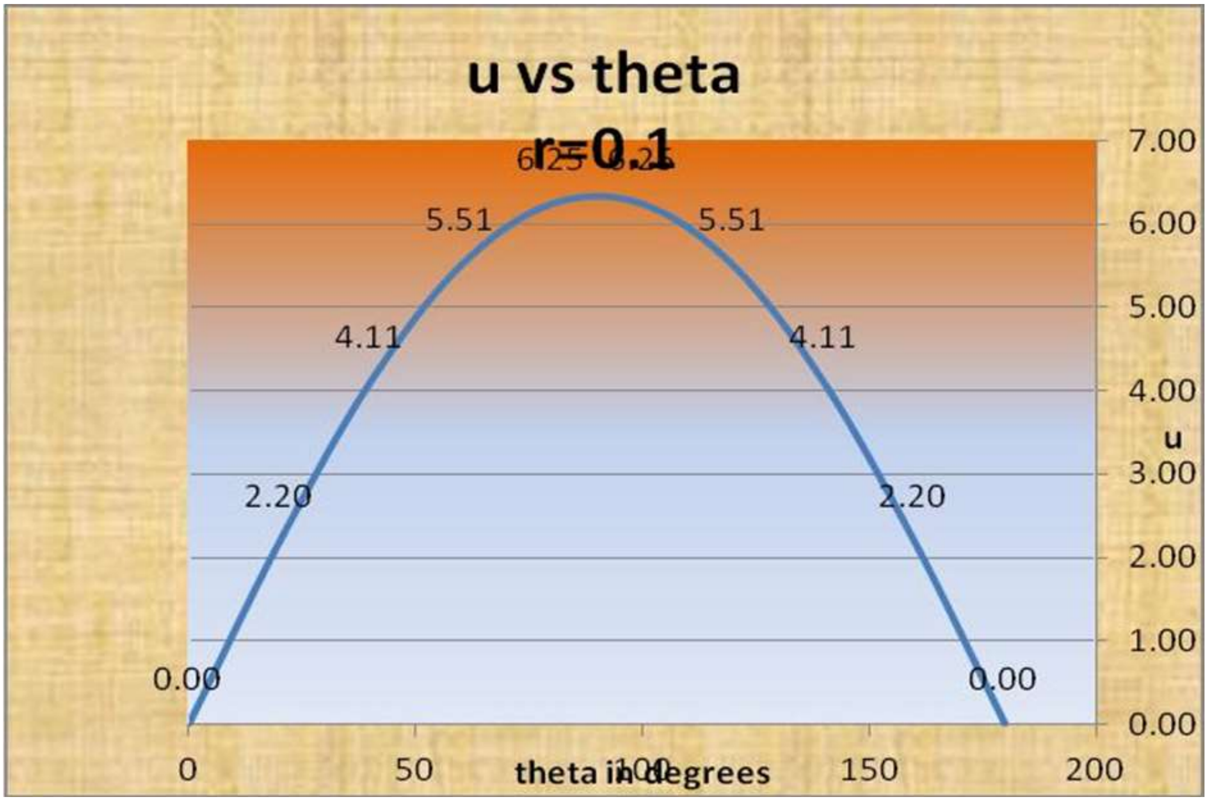
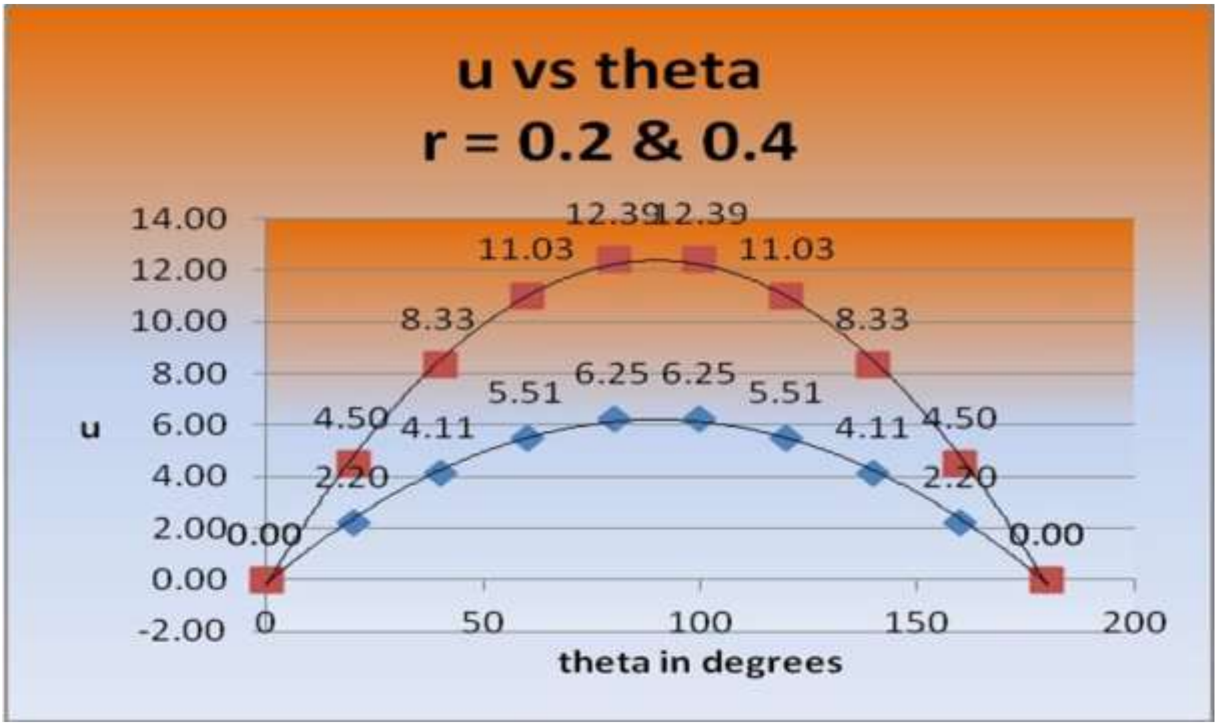

u vs theta
r=0.1
0.00
2.20
4.11
5.51
6.25
6.25
5.51
4.11
2.20
0.00
7.00
6.00
5.00
4.00
3.00
2.00
1.00
0.00
u
0
50
100
150
200
theta in degrees



II. RESULTS AND DISCUSSIONS

We have chosen n=2 in our calculations. However, by considering higher values for n, we can obtain better details of the behavior of heat transfer through the slab material.

For n= 2 , r = 0.2 and r = 0.4 the plots are presented in figs nos. 1 &2. We observe the variations of the temperature from 0 deg. to 6.25 deg in r = 0.2 while for r = 0.4 , temperature varies from 0 to 12.39 showing the expected heat transfer behavior in the slab. The calculations are made for all other cases i.e. r = 0.6, 0.8... up to 2.0.

The temperature distribution $u(r,\theta)$ is satisfactory in the semi-circular lamina which is insulated laterally.

The results are in good comparison with the results obtained by ANSYS V 12.0.

T = 50 C , a = 2 m,
r = 0.2

θ	$u(r,\theta)$
0	0
20	2.195745
40	4.110491
60	5.513289
80	6.251103
100	6.251103
120	5.513289
140	4.110491
160	2.195745
180	0

T=50 C, a = 2m,
r = 0.2 & 0.4

θ	$u(r,\theta)$	$u(r,\theta)$
0	0.00	0.00
20	2.20	4.50
40	4.11	8.33
60	5.51	11.03
80	6.25	12.39
100	6.25	12.39
120	5.51	11.03
140	4.11	8.33
160	2.20	4.50
180	0.00	0.00

The work can be extended to three dimensions i.e. (r,θ,z) to improve the accuracy in the temperature distribution profiles.

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